THE SEESAW TENSOR: TRANSLATED VACUA, GLUING AND THE UNIT

Prerequisites. From Chapter 46: the cuts $\mathrm{Cut}_j(\xi)$, their multiplicities mult_j and the seesaw tensor ss_ξ (Definitions 46.3.2 and 46.4.3), the generator ideles ϖ_ξ, ϖ_M (Definition 46.2.6), Problems 46.3.4 and 46.3.5, and Theorem 46.4.1. From Chapter 45: the join $\boxtimes$, the zero term zt, the stacked map A and the evaluated pair datum $\phi_i^{\mathrm{ev}}(\xi, y)$ (Definition 45.4.3). From Chapter 43: the translations $\mathfrak{t} \in \mathcal{T}$, the translated vacua $v_j(\mathfrak{t})$, the data $\mathbf{1}_j(\xi)$ of the lower chain, the levels K_j^X, the restriction res, chain data and the composite lift $B_{\vartheta,\Psi}$ (Section 43.4), and Problems 43.3.3, 43.5.4, 43.5.5. In a tensor over pairs, later pairs stand on the left. From Chapter 12: theta lines and intertwiners (Problems 12.2.1, 12.3.4, 12.5.2, Theorem 12.4.1). From Chapter 34: the frames of the spaces, Q_ξ, $\mathrm{Nm}\, Q_\xi$, the integral vacuum and c_{vac} (Sections 34.3, 34.6, 34.8), and the ring $\mathcal{O}^{[q]}$ of the $z \in \mathbb{C}$ with $|\eta(z)|_\ell \le 1$ for every prime $\ell \nmid q$ and every ring map $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$. The Weil representation over the finite adeles in its Schrödinger and lattice models [We], [MVW, Ch. 2]. As in Chapter 45, $\mathbf{1}_j(\xi)$ is the finite datum of the lift $Y_{j-1} \to Y_j$; ω_j^X is the finite Weil action on the data of the lift $X_j \to X_{j+1}$.

This chapter is the last of the four that prove the seesaw identity, Theorem 43.4.1, which Theorem A uses. Theorem 46.4.1 gives it on automorphic forms, for ramified source forms, with the tensor ss_ξ, a weighted sum over the cuts of evaluated pair data, in place of a chain datum. Which scalar and which common denominator suffice to express ss_ξ in glued vacuum tensors $\mathrm{gl}(\mathfrak{t})$? Three local facts answer this: Fourier inversion on a residue module, the lattice model of the vacuum, and a neighbour lattice at N. One complex unit then works for all twists.

Throughout, ξ is a tame character and $q := \mathrm{Nm}\, Q_\xi$. A finite adelic vector is *integral away from* q if its entries are integral at every finite place not above a prime dividing q.

47.1. Residue modules and Fourier inversion

Definition 47.1.1. Let $(w_\gamma)_{\gamma \in \mathcal{Z}}$ be vectors of a complex vector space, $\mathfrak{d} \ge 1$ an integer and $\mathfrak{s} \ge 0$ real. A vector z is a *combination of the* w_γ, $\gamma \in \mathcal{Z}$, *with denominator* $\mathfrak{d}$ *and size* $\mathfrak{s}$ if $z = \sum_{\gamma \in \mathcal{Z}} r_\gamma w_\gamma$ with finitely many nonzero $r_\gamma \in \mathbb{C}$, $\mathfrak{d}\, r_\gamma \in \mathcal{O}^{[q]}$ for every γ and, if $Q_\xi = \varnothing$, $\sum_\gamma |\sigma(r_\gamma)| \le \mathfrak{s}$ for every $\sigma \in \mathrm{Aut}\, \mathbb{C}$.

Let $\mathfrak{m} \subseteq \mathcal{O}_E$ be a nonzero ideal, $\mathsf{A}_\mathfrak{m} := (\mathcal{O}_E/\mathfrak{m})^n$, $\psi_\mathfrak{m}$ a primitive additive character of $\mathcal{O}_E/\mathfrak{m}$, $\mathfrak{y} \cdot \mathfrak{b} := \sum_k \mathfrak{y}_k \mathfrak{b}_k$, and $\chi_\mathfrak{b}(\mathfrak{y}) := \psi_\mathfrak{m}(\mathfrak{y} \cdot \mathfrak{b})$. For the next problem alone, take the additive group $\mathbb{Z}/5\mathbb{Z}$ and characters with values in the fifth roots of unity of $\mathbb{F}_{11}$.

Problem 47.1.1. The class of 3 in $\mathbb{F}_{11}$ has order 5, and $\sum_{l=0}^{4} 3^{\,l(j-2)}$ is $0, 0, 5, 0, 0$ in $\mathbb{F}_{11}$ for $j = 0, \dots, 4$.

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At the places of $\mathfrak{m} := (\pi_M)\prod_{w\in Q_\xi} w$ (a modulus prime to its conjugate) the datum $\mathbf{1}_j(\xi)$ is the operator of the Schrödinger model attached to the point mass at the class $\mathfrak{i}\in \mathsf{A}_\mathfrak{m}$ of the inclusion matrix, $n=\dim Y_{j-1}\dim Y_j$, applied to the vacuum vac; at the other finite places it is the vacuum. Here $\psi_\mathfrak{m}(u \bmod \mathfrak{m})=\psi_0(\varpi_\xi^{-1}u)$ for integral finite adeles u, with ϖ_ξ the generator idele of $\mathfrak{m}$ of Definition 46.2.6 and ψ_0 the self-dual character of the finite adeles; a residue function is read on the reduction at the places of $\mathfrak{m}$, and is zero off the integral box there. The operator of $\chi_\mathfrak{b}$, on vectors fixed by the integral translations at the places of $\mathfrak{m}$ and their conjugates (as vac is), is the Heisenberg translation $\mathrm{Heis}(t_\mathfrak{b})$, where $t_\mathfrak{b}$ is the complex conjugate of the part at the places of $\mathfrak{m}$ of $\varpi_\xi^{-1}\tilde{u}_\mathfrak{b}$, $\tilde{u}_\mathfrak{b}$ any integral lift of $\mathfrak{b}$.

Problem 47.1.2. The point mass at $\mathfrak{i}$ is $|\mathsf{A}_\mathfrak{m}|^{-1}\sum_\mathfrak{b}\psi_\mathfrak{m}(-\mathfrak{i}\cdot\mathfrak{b})\,\chi_\mathfrak{b}$. Hence $\mathbf{1}_j(\xi)=|\mathsf{A}_\mathfrak{m}|^{-1}\sum_\mathfrak{b}\psi_\mathfrak{m}(-\mathfrak{i}\cdot\mathfrak{b})\,\mathrm{Heis}(t_\mathfrak{b})\mathrm{vac}$, and for every $\sigma\in\mathrm{Aut}\,\mathbb{C}$ the absolute values of the σ-conjugates of these coefficients sum to 1.

Problem 47.1.3. Each $\bar\pi_M t_\mathfrak{b}$ is integral away from q, and $\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^n$ divided by $|\mathsf{A}_\mathfrak{m}|$ is $q^{-n}\in\mathcal{O}^{[q]}$. So $\mathbf{1}_j(\xi)$ is a combination of the $\mathrm{Heis}(\mathfrak{z})\mathrm{vac}$, over the $\mathfrak{z}$ with $\bar\pi_M\mathfrak{z}$ integral away from q, with denominator $\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^n$ and size 1.

47.2. The lattice model

Let V be $\mathrm{Hom}(Y_{j-1},Y_j)$ or $\mathrm{Hom}(X_j,X_{j+1})$, with its standard lattice Λ in the finite adelic frame of Section 34.3, finite Heisenberg translations $\mathrm{Heis}(\mathfrak{z})$, Weil representation ω and vacuum vac. For background on the Weil representation and its models see [We], [MVW, Ch. 2]; we use the following, in the normalization of [GL]. (L1) $\mathrm{Heis}(\mathfrak{z})\mathrm{vac}=\mathrm{vac}$ for $\mathfrak{z}\in\Lambda$, and the vectors of the finite part fixed by all these translations are the multiples of vac (compare Problem 12.2.1). (L2) $\mathrm{Heis}(\mathfrak{z})\mathrm{Heis}(\mathfrak{z}')=\zeta\,\mathrm{Heis}(\mathfrak{z}+\mathfrak{z}')$, where ζ is a root of unity depending only on $\sum_k\bar{\mathfrak{z}}'_k\mathfrak{z}_k$ and equal to 1 when this sum is 0. (L3) $\omega(h)\mathrm{Heis}(\mathfrak{z})=\mathrm{Heis}(h\mathfrak{z})\,\omega(h)$ for $h\in U(V)(\mathbb{A}_f)$. (L4) $\omega(h)\mathrm{vac}=c_h\sum_{\mathfrak{z}\in\Lambda/(\Lambda\cap h^{-1}\Lambda)}\mathrm{Heis}(h\mathfrak{z})\mathrm{vac}$ with $c_h\neq 0$; compare the intertwiner defined in Section 12.3 and evaluated in Problem 12.3.4. The scalar c_h is the Weil multiplier divided by its squared absolute value. If h is integral away from q, then $|\eta(c_h)|_\ell=1$ for every prime $\ell\nmid q$ and every ring map $\eta\colon\mathbb{C}\to\mathbb{C}_\ell$; if h is integral at every finite place, then $c_h=1$. *Status*: we use the classical lattice model in the normalization of [GL], which proves (L1)–(L4) and the multiplier bounds.

Problem 47.2.1. Let $h\in U(V)(\mathbb{A}_f)$. A vector f of the finite part with $\mathrm{Heis}(h\mathfrak{z})f=f$ for all $\mathfrak{z}\in\Lambda$ is a multiple of $\omega(h)\mathrm{vac}$.

Problem 47.2.2. Let $h\in U(V)(\mathbb{A}_f)$ be integral away from q, and let $\bar\pi_M\mathfrak{z}_0$ be integral away from q. Then $\omega(h)\mathrm{Heis}(\mathfrak{z}_0)\mathrm{vac}$ is a combination of the $\mathrm{Heis}(\mathfrak{z})\mathrm{vac}$, over the $\mathfrak{z}$ with $\bar\pi_M\mathfrak{z}$ integral away from q, with denominator 1 and size 1.

The action ω_j on the data of the lower chain is that of Problem 47.2.2 times a determinant-character factor, which at points integral away from q has ℓ-adic norm 1 for $\ell\nmid q$ and absolute value 1 under every $\sigma\in\mathrm{Aut}\,\mathbb{C}$. So the moved data $\mathbf{1}_j(\xi)^y$ at points integral away from q keep the denominators of Problem 47.1.3 and size 1. If the two translations become integral away from q after multiplication by $\bar\pi_M$, the join of their translated vacua is a root of unity times a translated vacuum on $\mathrm{Hom}(Y_{2i+1},Y_{2i}^-\oplus Y_{2i+2})$, whose translation becomes integral away from q after multiplication by π_M on the first block and by $\bar\pi_M$ on the second. Evaluation

translates the joined datum by A_f, takes its finite zero term along the isotropic part I of the splitting, and applies $\boxtimes^{-1}$ on the definite spaces.

Problem 47.2.3. Let F be a nonarchimedean local field with integers $\mathcal{O}_F$, W a finite-dimensional F-vector space with a nondegenerate symmetric or alternating bilinear form, $L \subset W$ a lattice with $L = \{x : \langle x, L\rangle \subseteq \mathcal{O}_F\}$, and $I \subseteq W$ an isotropic subspace. Then the image of $L \cap I^{\perp}$ in $I^{\perp}/I$ is a lattice equal to its own dual.

Let L be the standard lattice of $\mathrm{Hom}(Y_{2i+1}, Y_{2i}^{-} \oplus Y_{2i+2})$ in its finite adelic frame. The proof identifies the reduced lattice with a finite adelic unitary image of the standard lattice in the quotient and deduces its self-duality for the pairing $(x, w) \mapsto \psi_0(\bar{x}^t w)$; Problem 47.2.3 is the analogue at one place. The image of $L \cap I^{\perp}$ in $I^{\perp}/I$, identified by the splitting with the finite adelic space of $\mathrm{Hom}(X_{2i+1}, X_{2i}^{-} \oplus X_{2i+2})$, is the *zero-term lattice* of pair i. The zero term of the vacuum on the first Hom space is a number $c_i' \neq 0$ times the sum of the translates of the vacuum on the second by representatives of the zero-term lattice modulo its intersection with the standard lattice there.

47.3. The neighbour at N

Which lattice is the zero-term lattice? Let P be the set of places $w \mid \pi_N$ and $\bar{P}$ the set of their conjugates; they are disjoint, with union the set of places above N. Let $\mathbf{1}_P$ be the finite adele equal to 1 on P and 0 elsewhere, π_P the finite idele equal to π_N on P, to $\bar{\pi}_N^{-1}$ on $\bar{P}$ and to 1 elsewhere, and u_{-1} a fixed unit of the integral finite adeles with $u_{-1}\bar{u}_{-1} = -1$. For a pair i let e, e'' be the last basis vectors of X_{2i} and X_{2i+2} in $X_{2i}^{-} \oplus X_{2i+2}$, and put $e_P := e + \bar{u}_{-1}\bar{\mathbf{1}}_P e''$. For columns write $z^* := \bar{z}^t$.

Problem 47.3.1. Over a commutative ring with involution, let z be a column with $z^*z = 1$, δ a scalar and $\gamma := 1 + (\delta - 1)zz^*$. Then $\gamma z = \delta z$, $\gamma z' = z'$ when $z^*z' = 0$, and $\gamma^*\gamma = 1$ if and only if $\delta\bar{\delta} = 1$. Moreover $e_P^* e_P = 1$ and $\pi_P \bar{\pi}_P = 1$.

Definition 47.3.1. The *neighbour lattice* of pair i is the image of the standard lattice of $\mathrm{Hom}(X_{2i+1}, X_{2i}^{-} \oplus X_{2i+2})$ under the matrix acting on the target as $\gamma_P := 1 + (\pi_P - 1)e_P e_P^*$ at the places above N, and at the other finite places as the identity for $i < \varrho$; for $i = \varrho$, at these places it scales the last row of X_m by $\varpi_M/\bar{\varpi}_M$ and fixes the other rows, ϖ_M the generator idele of (π_M) of Definition 46.2.6.

For a local example take $N = 5$ and the split algebra $\mathbb{Q}_5 \times \mathbb{Q}_5$, with factors indexed by $w \in P$ and $\bar{w}$ and conjugation exchanging them; take $\pi_N = (5, 1)$, $u_{-1} = (1, -1)$, $\mathbf{1}_P = (1, 0)$, and keep the rows e, e''.

Problem 47.3.2. In this model $e_P = \big((1,1),(0,1)\big)$, $\pi_P = (5, \frac{1}{5})$, and the components of γ_P at w and $\bar{w}$ are $G = \left(\begin{smallmatrix} 5 & 4 \\ 0 & 1 \end{smallmatrix}\right)$ and $G' = \left(\begin{smallmatrix} 1/5 & 0 \\ -4/5 & 1 \end{smallmatrix}\right)$, with $G^t G' = 1$.

To a residue modulo N at w, with representative $y \in \{0, \ldots, N-1\}$, the proof attaches the translation supported at $\bar{w}$ with entries $u_{-1}\bar{y}/N$ in row e and $-\bar{y}/N$ in row e'', where $\bar{y}/N$ is the conjugate of the adele equal to y/N at w and 0 elsewhere.

Problem 47.3.3. G is integral, and $G'\mathbb{Z}_5^2 + \mathbb{Z}_5^2 = \mathbb{Z}_5^2 + \mathbb{Z}_5 \cdot \frac{1}{5}(1,1)^t$; the five classes modulo $\mathbb{Z}_5^2$ are those of $\frac{1}{5}(y,y)^t$, $y = 0, \ldots, 4$. The translation attached to y is $-\frac{1}{5}(y,y)^t$ at $\bar{w}$, so one residue labels both rows.

Black box. For every pair i of the proof the zero-term lattice is the neighbour lattice; the lift of its columns is integral and orthogonal to the finite stacked map. We take the lattice equality and the two properties of the lift from [GL].

A representative of the neighbour lattice modulo its integral points is indexed by one residue modulo N for each basis vector of X_{2i+1} and each $w \in P$, specifying both rows e, e'', and one class away from N. Under the join on the definite spaces, Gl_i corresponds to the sum of the Heisenberg operators of these residue translations. Let C_i be c_i' times the number of classes away from N; it depends only on the data. The proof obtains: for n prime to N and $\mathfrak{z}$ on $\mathrm{Hom}(Y_{2i+1}, Y_{2i}^{-} \oplus Y_{2i+2})$ with π_M times its first block and $\bar{\pi}_M$ times its second integral away from n, the evaluation of $\mathrm{Heis}(\mathfrak{z})\mathrm{vac}$ is a finite sum of the vectors $C_i\,\mathrm{Gl}_i(\mathrm{Heis}(t_1)\mathrm{vac} \otimes \mathrm{Heis}(t_2)\mathrm{vac})$, with $\bar{\pi}_M t_1$, $\bar{\pi}_M t_2$ integral away from n, whose coefficients become integral away from n after multiplication by $\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^{(n_{2i}+n_{2i+2})n_{2i+1}}$ and have σ-conjugates of absolute sum at most 1 for every $\sigma \in \mathrm{Aut}\,\mathbb{C}$.

47.4. Invariance

Problem 47.4.1. Let sw exchange the coordinates of $\mathbb{Q}^2$. For $v = (1,2)^t$ and $v' = (3,-1)^t$, the sum of the matrices in the orbit of vv'^t under $X \mapsto \mathrm{sw}\,X\,\mathrm{sw}^t$ is $\left(\begin{smallmatrix} 1 & 5 \\ 5 & 1 \end{smallmatrix}\right)$ and is fixed by this action.

Problem 47.4.2. Let a group H act on finite sets $\mathcal{X}_0, \dots, \mathcal{X}_{m-1}$ and linearly on a complex vector space; let $\mathrm{mult}_j \colon \mathcal{X}_j \to \mathbb{Z}_{>0}$ be H-invariant and $\mathcal{E}$ a function on $\mathcal{X}_0 \times \cdots \times \mathcal{X}_{m-1}$ with $\mathcal{E}(bc) = b\,\mathcal{E}(c)$ for $b \in H$. Then $\sum_c \prod_j \mathrm{mult}_j(c_j)^{-1}\mathcal{E}(c)$ is fixed by H.

In the proof, let $b_j \in K_j^X$ for $j < m$ and $b_m := 1$. For $j < m$ the image of b_j in $U(Y_j)(\mathbb{A}_f)$ lies in the finite Bessel subgroup, permutes $\mathrm{Cut}_j(\xi)$ and keeps the multiplicities; and evaluation is equivariant: the evaluated pair datum at the translated points is $\omega_{2i}^X(b_{2i}, b_{2i+1}) \otimes \omega_{2i+1}^X(b_{2i+1}, b_{2i+2})$ applied to $\phi_i^{\mathrm{ev}}(\xi, y)$. By Problem 47.4.2, ss_ξ is fixed by $\bigotimes_j \omega_j^X(b_j, b_{j+1})$ for all such b.

47.5. Denominators and sizes

Definition 47.5.1. The *residue count* is $c_{\mathrm{res}} := M^{f n_m^2}$.

Since q is prime to N, the pair evaluation applies with $n = q$. Thus for $0 \le i \le \varrho$ and points y integral away from q, $\phi_i^{\mathrm{ev}}(\xi, y)$ is a combination of the vectors $C_i\,\mathrm{Gl}_i(\mathrm{Heis}(t_1)\mathrm{vac} \otimes \mathrm{Heis}(t_2)\mathrm{vac})$, over t_1, t_2 with $\bar{\pi}_M t_1$, $\bar{\pi}_M t_2$ integral away from q, with denominator $\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^{\mathfrak{e}_i}$ and size 1, where $\mathfrak{e}_i := \dim Y_{2i} \dim Y_{2i+1} + \dim Y_{2i+1} \dim Y_{2i+2} + (n_{2i} + n_{2i+2})n_{2i+1}$. The proof states this with the uniform denominator c_{res}^4.

Problem 47.5.1. Substituting $1, 2, 3$ for the three Y-dimensions and for the three X-ranks in $\mathfrak{e}_i$ gives $16 \le 4 \cdot 3^2$. For the ranks of the proof, $\mathfrak{e}_i \le 4n_m^2$, so $\#(\mathcal{O}_E/\pi_M\mathcal{O}_E)^{\mathfrak{e}_i}$ divides c_{res}^4.

Problem 47.5.2. Let z be a combination of the w_γ, $\gamma \in \mathcal{Z}$, with denominator $\mathfrak{d}$ and size $\mathfrak{s}$, and let each w_γ be a combination of the $w'_{\gamma'}$, $\gamma' \in \mathcal{Z}'$, with denominator $\mathfrak{d}'$ and size $\mathfrak{s}'$. Then z is a combination of the $w'_{\gamma'}$ with denominator $\mathfrak{d}\mathfrak{d}'$ and size $\mathfrak{s}\mathfrak{s}'$. If z_k is a combination of the $w_{k,\gamma}$ with denominator $\mathfrak{d}_k$ and size $\mathfrak{s}_k$ $(k < l)$, then $z_{l-1} \otimes \cdots \otimes z_0$ is a combination of the $w_{l-1,\gamma_{l-1}} \otimes \cdots \otimes w_{0,\gamma_0}$ with denominator $\prod_k \mathfrak{d}_k$ and size $\prod_k \mathfrak{s}_k$.

The integral vacuum of $\mathrm{Hom}(X_j, X_{j+1})$ is $c_{\mathrm{vac}}^{n_j n_{j+1}}\mathrm{vac}$ (Section 34.8). Put

$$c_{\mathrm{gl}} := \prod_{i=0}^{\varrho} \frac{C_i}{c_{\mathrm{vac}}^{n_{2i}n_{2i+1}}\, c_{\mathrm{vac}}^{n_{2i+1}n_{2i+2}}},$$

a nonzero number. By Problems 47.5.1 and 47.5.2, for points y integral away from q the tensor $\bigotimes_{i=0}^{\varrho}\phi_i^{\mathrm{ev}}(\xi, y)$ is a combination of the $c_{\mathrm{gl}}\,\mathrm{gl}(\mathfrak{t})$, over the $\mathfrak{t}$ with every $\bar\pi_M\mathfrak{t}_j$ integral away from q, with denominator c_{res}^{2m} and size 1.

Problem 47.5.3. Each $\mathrm{mult}_j(c)^{-1}$, $c \in \mathrm{Cut}_j(\xi)$, lies in $\mathcal{O}^{[q]}$, and if $Q_\xi = \varnothing$, then $\sum_{c_0,\dots,c_{m-1}}\prod_j \mathrm{mult}_j(c_j)^{-1} \le c_{\mathrm{res}}^m$. Hence ss_ξ is a combination of the $\bigotimes_i \phi_i^{\mathrm{ev}}(\xi, c)$, over tuples c of cut points with representatives integral away from q, with denominator 1 and size c_{res}^m.

The pair evaluation, the tensor bound above and Problem 47.5.3, followed by substitution (Problem 47.5.2), give the following theorem.

Theorem 47.5.1. *There is $c_{\mathrm{gl}} \in \mathbb{C}$, depending only on the data, such that for every tame character ξ, with $q = \mathrm{Nm}\,Q_\xi$, the tensor ss_ξ is a combination of the $c_{\mathrm{gl}}\,\mathrm{gl}(\mathfrak{t})$, over the $\mathfrak{t} \in \mathcal{T}$ with every $\bar\pi_M\mathfrak{t}_j$ integral away from q, with denominator c_{res}^{2m} and size c_{res}^m: there are $r_{\xi,\mathfrak{t}} \in \mathbb{C}$, finitely many nonzero, with*

$$\mathrm{ss}_\xi = \sum_{\mathfrak{t}} r_{\xi,\mathfrak{t}}\, c_{\mathrm{gl}}\,\mathrm{gl}(\mathfrak{t}), \qquad c_{\mathrm{res}}^{2m} r_{\xi,\mathfrak{t}} \in \mathcal{O}^{[q]},$$

$$Q_\xi = \varnothing \implies \sum_{\mathfrak{t}} |\sigma(r_{\xi,\mathfrak{t}})| \le c_{\mathrm{res}}^m \text{ for all } \sigma \in \mathrm{Aut}\,\mathbb{C}.$$

47.6. The unit and the seesaw identity

By Theorem 46.4.1 and Theorem 47.5.1, for every ramified source form φ the composite of the lower lifts on automorphic forms applied to $\mathcal{P}_{g,\lambda}\varphi$ is $c'c_{\mathrm{gl}}/c_{\mathrm{res}}^{2m}$ times $\mathrm{res}\,\Theta_\xi^X\big(\sum_{\mathfrak{t}} c_{\mathrm{res}}^{2m} r_{\xi,\mathfrak{t}}\mathrm{gl}(\mathfrak{t})\big)\varphi$, and these coefficients lie in $\mathcal{O}^{[q]}$. The scalar may be 0.

Problem 47.6.1. Let $c' = 2$, $c_{\mathrm{gl}} = 3$, denominator 4 and $r = (\frac14, -\frac12)$. Applying Problem 43.5.4 to the scalar $c'c_{\mathrm{gl}}/4$ gives $u = \frac23$ and $c = u^{-1} = \frac32$. The cleared vector $\Psi = 4r$ is $(1, -2)$ and satisfies $c\Psi = c'c_{\mathrm{gl}}r = (\frac32, -3)$. If $c' = 0$, the same identity holds with $c = 1$ and $\Psi = 0$.

Problem 47.6.2. Let $\mathcal{V}$ be the set of finite places above the primes dividing Mq, and $\mathbf{1}_{\mathcal{V}}\mathfrak{t} := (\mathbf{1}_{\mathcal{V}}\mathfrak{t}_j)_j$. If every $\bar\pi_M\mathfrak{t}_j$ is integral away from q, then $v_j(\mathbf{1}_{\mathcal{V}}\mathfrak{t}) = v_j(\mathfrak{t})$ for every j, so $\mathrm{gl}(\mathbf{1}_{\mathcal{V}}\mathfrak{t}) = \mathrm{gl}(\mathfrak{t})$.

Problem 47.6.3. Let the tame character ξ be a twist ϑ, with the nonzero c_{gl} chosen above, coefficients $r_{\vartheta,\mathfrak{t}}$ for this choice in Theorem 47.5.1, and $\mathcal{V}$ as in Problem 47.6.2 for $q = \mathrm{Nm}\,Q_\vartheta$. If $c' \neq 0$, put $\Psi(\mathfrak{t}) := c_{\mathrm{res}}^{2m}\sum r_{\vartheta,\mathfrak{t}'}$ over the $\mathfrak{t}'$ with $\mathbf{1}_{\mathcal{V}}\mathfrak{t}' = \mathfrak{t}$; otherwise $\Psi := 0$. Then Ψ is a chain datum for ϑ; for $Q_\vartheta = \varnothing$ its size is at most $c_{\mathrm{res}}^{2m}c_{\mathrm{res}}^m = M^{3fmn_m^2}$.

Theorem 46.4.1, Theorem 47.5.1, the invariance of ss_ξ, Problems 43.5.4, 47.6.2 and 47.6.3, and the passage from automorphic to algebraic forms on Y_m prove Theorem 43.4.1, with $c = c'c_{\mathrm{gl}}/c_{\mathrm{res}}^{2m}$ if $c' \neq 0$ and $c = 1$, $\Psi = 0$ otherwise (compare Problem 43.5.5). Passage to algebraic forms on Y_m carries the restriction on the right of the automorphic identity to the restriction in $B_{\vartheta,\Psi}$. Nothing is said about the size of c.

Definition 47.6.1. For $\mathfrak{t} \in \mathcal{T}$ let $\mathrm{pl}_0(\mathfrak{t}) := 1 \in \mathbb{C}$ and $\mathrm{pl}_{i+1}(\mathfrak{t}) := \big(v_{2i}(\mathfrak{t}) \otimes v_{2i+1}(\mathfrak{t})\big) \otimes \mathrm{pl}_i(\mathfrak{t})$ for $0 \le i \le \varrho$. The *plain vacuum tensor* is $\mathrm{pl}(\mathfrak{t}) := \mathrm{pl}_{\varrho+1}(\mathfrak{t})$.

If every $\mathfrak{t}_j$ vanishes at the places above N, then $\mathrm{gl}(\mathfrak{t})$ is the sum of the $\mathrm{pl}(\mathfrak{t} + \mathfrak{g}_y)$ over the residues y of the gluings, $\mathfrak{g}_y$ the tuple of their translations; Chapter 48 uses this expansion of the composite lift.

We have proved Theorem 47.5.1 and with it the seesaw identity. Taken as given: the Schrödinger and lattice models with the multiplier bounds, and the zero-term lattice, its lift and the pair evaluation quoted above.
In the Lean formalization. Declaration names omit the prefix `New.`.
Theorem 47.5.1 is `Data.exists_isAdmissibleCombination_seesawDatum` in the Lean proof [GL]. Definition 47.1.1 specializes `Seesaw.IsAdmissibleCombination` to $q = \mathrm{Nm}\, Q_\xi$, the size condition $Q_\xi = \varnothing$, positive denominators and nonnegative bounds. Definition 47.5.1 is `Data.residueMatrixCount`; Definition 47.6.1 and $\mathrm{gl}(\mathfrak{t})$ are `Data.ungluedVacuum` and `Data.gluedVacuum`. The uniform bound for one evaluated pair is `Data.exists_isAdmissibleCombination_evaluatedDatum`, the bound for their tensor
`Data.exists_isAdmissibleCombination_gluedVacuum`, and joint invariance `Data.isInvariant_seesawDatum`. The zero-term lattice equality is `Seesaw.DoubleLiftData.zeroTermLattice_splitting_eq`. Theorem 43.4.1 is `Data.seesaw_identity`. The other statements read from the proof are, in [GL] (paths relative to `grh/src/`): for the residue operator
`Lib/Weil/DualPair/Defs.lean` (587, 625) and
`New/UpperBound/Objects/Rows.lean` (377); for (L1)–(L4)
`Lib/Weil/Construction/VacuumProjections.lean` (102, 203, 450, 755, 928), with the multiplier bounds in
`New/UpperBound/Seesaw/AdmissibleCombinations.lean` (451) and
`Lib/Weil/Construction/FiniteMultiplier.lean` (948, 979). In
`New/UpperBound/Seesaw/`: the moved and joined data `GoodDatum.lean` (267, 370); the evaluation, the reduced lattice and the equivariance `EdgePairEvaluation.lean` (487, 514, 853, 988); the zero term of the vacuum, the lift and the pair evaluation `AdmissibleEvaluation.lean` (248, 329, 565, 596); the residue translations and representatives `NeighbourLattice.lean` (171, 996); the gluing `NeighbourVacuum.lean` (241); the rank bound `GoodDatum.lean` (421); the inputs to invariance `Cut.lean` (173, 182, 199). The plain expansion is in `New/UpperBound/Divisibility/ChainDatumExpansion.lean` (233).

Hints

§47.1.

47.1.2: Orthogonality of the characters of $\mathsf{A}_{\mathfrak{m}}$; the operator depends linearly on the function. Compare Problem 43.3.3.

47.1.3: $\mathfrak{m}$ is the product of the coprime ideals (π_M) and $\prod_{w \in Q_\xi} w$.

§47.2.

47.2.1: Move the problem to the standard lattice.

47.2.2: (L2)–(L4). If $Q_\xi = \varnothing$, then $h\Lambda = \Lambda$.

47.2.3: Compute the annihilator of $L \cap I^\perp$ in W; $L \cap I^\perp$ is a direct summand of L.

§47.3.

47.3.1: Expand $\gamma^*\gamma$. $\mathbf{1}_P\bar{\mathbf{1}}_P = 0$.

§47.4.

47.4.2: Substitute $c \mapsto bc$.

§47.5.

47.5.1: All ranks are at most n_m, and $\#(\mathcal{O}_E/\pi_M\mathcal{O}_E) = M^f$.

47.5.2: $\mathcal{O}^{[q]}$ is a ring; the triangle inequality gives the sizes.

47.5.3: Problems 46.3.4 and 46.3.5.

§47.6.

47.6.2: Split $\mathfrak{t}_j$ into its parts on $\mathcal{V}$ and off $\mathcal{V}$. Which part fixes the vacuum, and what is their Heisenberg phase?

47.6.3: Problems 47.6.2 and 47.4.2; regroup equal indices.

References

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