LATTICES, CUTS AND THE RAMIFIED FORMS

Prerequisites. Lattices and their duals [OM]. From Chapter 45: stage functions $\mathcal{W}_j$ (Definition 45.1.1), the evaluated pair data $\phi_i^{\mathrm{ev}}(\xi, y)$ (Definition 45.4.3), the transposes $\Theta_{\xi,j}^{\top}$ with $\int z\,\Theta_{\xi,j} f = \int (\Theta_{\xi,j}^{\top} z) f$, the ordered pair tensors $\Phi_{\varrho} \otimes \cdots \otimes \Phi_0$, Problem 45.1.2 and Theorem 45.5.1; from Chapter 44, the grades (Definition 44.2.2) and the string maps $\mathrm{st}_j \colon Y_{j-1} \to Y_j$ (Definition 44.2.3). From Section 43.4: the lower and definite chains, their data, the definite lift $\Theta_{\xi}^{X}(\cdot)$ of a tensor, its Weil action ω^X, and the restriction res; from Chapter 43, Problems 43.3.2, 43.5.2 and 43.5.3. $K_j^{Y\circ}$ is the lattice stabilizer on Y_j, $K_j^Y(\xi)$ its subgroup reducing to 1 at the places dividing π_M and those of Q_ξ. K_M is the principal level of Y_0 (Sections 34.4 and 43.4); $K_0^Y(\xi) \subseteq K_M \subseteq K_0^{Y\circ}$. From Chapter 36: the Bessel subgroups (Section 36.1). The Poincaré map $\mathcal{P}_{g,\lambda}$ is that of Section 43.4. From Chapter 34: the spaces, $S \supseteq S_{\mathrm{ram}}$, T_w, z_w, tame characters ξ with Q_ξ and $\mathrm{Nm}\, Q_\xi$. R_j is the Bessel subgroup of Y_j, $[R_j] := R_j(F)\backslash R_j(\mathbb{A})$; for $0 \le j < m$ the string has length $m-1-j$, and $R_m = U(Y_m)$. $\varrho = m/2 - 1$. "The proof" is the one in [GL].

The seesaw identity (Theorem 43.4.1) is proved by pairing $\Theta_{\xi,m} \cdots \Theta_{\xi,1} \mathcal{P}_{g,\lambda}\varphi$ with a form on Y_m and transferring the Bessel period up the lower chain. Chapter 45 computes what is left after the transfers as a function of the outer variables $(g_k, y_k) \in U(Y_k)(\mathbb{A})$. At stage zero the finite sum is over classes in K_M. At the subsequent stages the Bessel periods must be integrated over the remaining outer variables. Which finite sets of level classes suffice to compute these integrals, and with which weights? The answer uses the lattices of the finite points and, at one place, the hypothesis $\varphi \in S_{\mathrm{ram}}$. The result is Theorem 46.4.1.

46.1. The first lift

Problem 46.1.1. Let y and z' be functions on $[U(Y_0)]$, fixed by right translation by K_M and by $K_0^Y(\xi)$ respectively, with yz' integrable. Then

$$\int_{[U(Y_0)]} y(h) \sum_{c \in K_M / K_0^Y(\xi)} z'(hc)\, dh = [K_M : K_0^Y(\xi)] \int_{[U(Y_0)]} y\, z'.$$

Hence, for every tame ξ, every $\varphi \in S$ and every form z on Y_1 as in Problem 45.1.2,

$$\int_{[U(Y_1)]} z\, \Theta_{\xi,1} \mathcal{P}_{g,\lambda}\varphi = [K_0^{Y\circ} : K_M] \sum_{c \in K_M / K_0^Y(\xi)} \int_{R_1(F)\backslash U(Y_1)(\mathbb{A})} \mathcal{W}_1\, z,$$

where the summand uses $(g, \tilde{c})$ in outer slot 0, with $\tilde{c} \in K_M$ any representative of c, and the identity in every other slot.

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46.2. Lattices of finite points

The lattice of the finite point enters the support condition. The classes used for the evaluated data are taken modulo $K_j^Y(\xi)$. For $0 \le j \le m-1$ write X_j for the definite part of Y_j and $X_j^\perp$ for the span of its string, both in the rational frame of Chapter 34.

Definition 46.2.1. The *lattice* $\Lambda(x)$ of a point $x \in U(Y_j)(\mathbb{A}_f)$ is the set of finite adelic vectors whose coordinates in the frame of x (the frame of Chapter 34 followed by x) are integral, so that $\Lambda(x) = x\Lambda(1)$ when x acts in that frame. Its *string parts* are $\Lambda^-(x) := \Lambda(x) \cap X_j^\perp$ and $\Lambda^+(x) := \operatorname{pr}\Lambda(x)$, where pr is the orthogonal projection to $X_j^\perp$. We write $\Lambda_j^\pm$ when the stage must be named.

Definition 46.2.2. A lattice is *graded* if it contains the part of each grade of each of its vectors.

As the frame of x is orthonormal, $\Lambda(x)$ is self-dual. The standard lattice $\Lambda(1)$ is graded. At a place $w \mid \pi_N$, the standard lattice of an odd string is spanned, in the last definite coordinate and the middle string coordinate, by vectors $(1, \epsilon/N)$ and $(0, \epsilon')$ with units ϵ, ϵ'. In the toy $N = 5$, the place has degree one, $\epsilon = \epsilon' = 1$. In $\mathbb{Q}_5^2 = \mathbb{Q}_5 e_1 \oplus \mathbb{Q}_5 e_2$ call e_1 the definite coordinate and e_2 the string coordinate, and define $\Lambda^\pm$ as in Definition 46.2.1 with $X^\perp = \mathbb{Q}_5 e_2$; no form is fixed here.

Problem 46.2.3. For $\Lambda = B\mathbb{Z}_5^2$, $B = \left(\begin{smallmatrix} 1 & 0 \\ 1/5 & 1 \end{smallmatrix}\right)$: $\Lambda^- = \mathbb{Z}_5 e_2$, $\Lambda^+ = 5^{-1}\mathbb{Z}_5 e_2$ and $\#\Lambda^+/\Lambda^- = 5$. For $\Lambda = D'\mathbb{Z}_5^2$, $D' = \operatorname{diag}(5, 1/5)$: $\Lambda^- = \Lambda^+ = 5^{-1}\mathbb{Z}_5 e_2$.

Problem 46.2.4. Over a DVR, or a product of two exchanged by the involution, let Λ be a lattice, self-dual for a nondegenerate hermitian form, in a space $W \perp W'$. Then the dual of $\Lambda \cap W$ in W is the projection of Λ to W. Hence, for $x, x' \in U(Y_j)(\mathbb{A}_f)$ and a finite idele ϖ, $\varpi\Lambda^-(x) \subseteq \Lambda^-(x')$ implies $\bar\varpi\Lambda^+(x') \subseteq \Lambda^+(x)$.

Problem 46.2.5. A function on $\mathbb{Z}/25$ invariant under translation by $5\mathbb{Z}/25$ has Fourier transform supported on $5\mathbb{Z}/25$.

Definition 46.2.6. Let ϖ_ξ and ϖ_M be the generator ideles fixed in the proof for the ideals $(\pi_M)\prod_{w\in Q_\xi} w$ and (π_M); each has component 1 away from the places dividing its ideal. The *idele of the modulus* is $\mathfrak{u} := \varpi_\xi/\varpi_M$. It is a uniformizer at every $w \in Q_\xi$ and a unit at every other finite place.

In this sheet $\mathfrak{a} := \varpi_W/\bar\varpi_M$, where ϖ_W is the self-dual scale of the Weil data. Throughout this paragraph let $0 \le j \le m-2$. The finite datum of lift $j+1$ is fixed by the finite Heisenberg translations by $\varpi_\xi v$ for every integral vector v in the finite space of maps from Y_j to Y_{j+1}. The proof shows: let $\mathcal{W}_{j+1}(g_\infty, x') \ne 0$, so that the integrand defining it (Definitions 44.4.2 and 45.1.1) is nonzero at some $r \in R_j(\mathbb{A})$, and let x be the finite part of $r(g_j, y_j)$; if $r_1 \in R_j(\mathbb{A}_f)$ and $r_2 \in R_{j+1}(\mathbb{A}_f)$ make $\Lambda(r_1 x)$ and $\Lambda(r_2 x')$ graded, then

$$(\bar{\mathfrak{u}}/\mathfrak{a})\,\mathrm{st}_{j+1}\Lambda_j^+(r_1 x) \subseteq \Lambda_{j+1}^-(r_2 x').$$

For the standard lattices it shows $\mathrm{st}_{j+1}\Lambda_j^+(1) = \mathfrak{a}\,\Lambda_{j+1}^-(1)$.

Problem 46.2.7. Use the string matrices $J_3 = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, $J_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathrm{st}_1 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \end{smallmatrix}\right)$ of Chapter 45 ($N = 5$, string constant 1), and let $\hat{\mathcal{O}}$ be the integral finite adeles of $\mathbb{Q}(\mathrm{i})$. On the first string let $L^- = \hat{\mathcal{O}}^3$ and $L^+ = \operatorname{diag}(1, \frac{1}{5}, 1)\hat{\mathcal{O}}^3$, on the

second $L' = \hat{\mathcal{O}}^2$. Then L^- and L^+ are dual for J_3, L' is self-dual for J_2, and $\mathrm{st}_1 L^+ = L'$.

Definition 46.2.8. A point $x \in U(Y_j)(\mathbb{A}_f)$ has the *lattice bounds* at stage j if $\bar{\mathfrak{u}}^j \Lambda_j^-(1) \subseteq \Lambda_j^-(x)$ and $\mathfrak{u}^j \Lambda_j^-(x) \subseteq \Lambda_j^-(1)$.

Problem 46.2.9. Let $0 \le j \le m-2$, and suppose $\mathfrak{u}^j \Lambda_j^-(r_1 x) \subseteq \Lambda_j^-(1)$ and $(\bar{\mathfrak{u}}/\mathfrak{a})\,\mathrm{st}_{j+1}\Lambda_j^+(r_1 x) \subseteq \Lambda_{j+1}^-(r_2 x')$. Then $\bar{\mathfrak{u}}^{j+1}\Lambda_{j+1}^-(1) \subseteq \Lambda_{j+1}^-(r_2 x')$.

Definition 46.2.10. $\Lambda(x)$ is *split* at a finite place w if $1_w \Lambda^+(x) \subseteq \Lambda^-(x)$, where 1_w is the idempotent finite adele that is 1 at w and 0 elsewhere.

The proof shows that $\Lambda_j^+(1) = \Lambda_j^-(1)$ for an even string, and that for an odd string the two differ only in the middle coordinate, by a simple pole at the places above N: at each $w \mid N$ the quotient is the residue field k_w, as in Problem 46.2.3.

Problem 46.2.11. Let $1 \le j \le m-1$ and $\bar{\mathfrak{u}}^j \Lambda_j^-(1) \subseteq \Lambda_j^-(x)$, but $\mathfrak{u}^j \Lambda_j^-(x) \not\subseteq \Lambda_j^-(1)$. Then some $\mathfrak{u}^j v$, $v \in \Lambda_j^-(x)$, lies in $\Lambda_j^+(1)$ but not in $\Lambda_j^-(1)$. Moreover, the string of Y_j is odd, j is even, and $\Lambda(x)$ is split at a place w dividing π_N.

Problem 46.2.12. For $z = \mathrm{diag}(2,1)$, the scaling of the definite coordinate by a generator of $(\mathbb{Z}/5)^\times$: $B^{-1} z B = \left(\begin{smallmatrix} 2 & 0 \\ -1/5 & 1 \end{smallmatrix}\right)$ is not integral, and $D'^{-1} z D' = z$.

In general, if $\Lambda(x)$ is split at $w \mid \pi_N$, then $x^{-1} z_w x$, with z_w acting as a scalar on X_j and trivially on the string, is integral and 1 off $w, \bar{w}$. It belongs to $K_j^Y(\xi)$.

Problem 46.2.13. For $\mathrm{sh}\, e_k = e_{k+1 \bmod 4}$ on $\mathbb{C}^4$: $(1 - \mathrm{sh})e_0 = (1,-1,0,0)$ and $(1 + \mathrm{sh} + \mathrm{sh}^2 + \mathrm{sh}^3)(1 - \mathrm{sh})e_0 = 0$.

Problem 46.2.14. Suppose $\omega^X(z_w, z_w)$ acts trivially on the data of every lift $X_k \to X_{k+1}$, and $\omega^X(1, z_w)$, acting on one factor alone, fixes a tensor v of data. Then the (partial) chain $\Theta_\xi^X(v)$ satisfies $\Theta_\xi^X(v)\, T_w = \Theta_\xi^X(v)$, and it vanishes on S_{ram}.

For $w \mid \pi_N$, the diagonal scaling $\omega^X(z_w, z_w)$ is the identity on every lift: the base changes of the characters of the lifts are unramified at the places above N. For $0 \le i < \varrho$, embed $U(X_{2i+2})$ into $U(Y_{2i+2})$ by the identity on the string. If $y_{2i+2}^{-1} z_w y_{2i+2} \in K_{2i+2}^Y(\xi)$, then $\omega^X(1, z_w)$ on the second tensor factor fixes $\phi_i^{\mathrm{ev}}(\xi, y)$.

Problem 46.2.15. Let $\varphi \in S_{\mathrm{ram}}$, $2 \le j \le m-1$ even, and $\Lambda(x)$ split at some $w \mid \pi_N$. Then $\mathcal{W}_j(\gamma(g_\infty, x)) = 0$ for every $\gamma \in R_j(\mathbb{A})$ and every archimedean point g_∞.

Problem 46.2.16. Let $\varphi \in S_{\mathrm{ram}}$, $y_0 \in K_M$, $1 \le j \le m-1$, and $\mathcal{W}_j(g_\infty, x) \ne 0$. Then $r_2 x$ has the lattice bounds at stage j for every $r_2 \in R_j(\mathbb{A}_f)$ with $\Lambda(r_2 x)$ graded.

This is where the hypothesis $\varphi \in S_{\mathrm{ram}}$ enters. We use two facts for the field E of the data, which contains a primitive 5040th root of unity. (i) Every finite point x has a translate ux with u in the unipotent subgroup of $R_j(\mathbb{A}_f)$ and $\Lambda(ux)$ graded. This is the lattice form of the Iwasawa decomposition [Ti, 3.3.2]. (ii) There is a finite set $\mathcal{Z}_j$ of finite points equal to 1 outside $Q_\xi \cup \bar{Q}_\xi$ such that, if the $R_j(\mathbb{A}_f)$-orbit of x contains a graded point satisfying the bounds, then $rx \in \mathcal{Z}_j K_j^{Y\circ}$ for some $r \in R_j(\mathbb{A}_f)$. We use (ii) as proved in [GL]. Two classical lattice facts used there are that only finitely many lattices lie between two fixed full lattices, and that a lattice is determined by its local components [OM].

46.3. Cuts and multiplicities

In this sheet let a group G_0 act on a set, let $G_1 \subseteq G_0$ be a subgroup and Cut a finite G_1-stable subset, and put $\mathrm{mult}(t) := \#\{rG_1 \in G_0/G_1 : r^{-1}t \in \mathrm{Cut}\}$, with the value 0 for an infinite count. For a finite group acting on a finite set, if the cut meets every orbit, this is the setting of Problem 43.3.2 with translates counted modulo G_1.

Problem 46.3.1. Let $G_0 = H = S_3$ act on H/H_0, $G_1 = H_0 := \langle(1\,2)\rangle$, $x = (1\,2\,3)$, $\mathrm{Cut} := \{hxH_0 : h \in H_0\} \subset H/H_0$. Then $\#\mathrm{Cut} = 2$, the multiplicities of the three cosets are $2, 2, 2$, and for $f_1(hH_0) := h(3)$,

$$\sum_{t\in H/H_0} f_1(t) = 6 = \sum_{rH_0\in H/H_0}\ \sum_{c\in\mathrm{Cut}} \frac{f_1(rc)}{\mathrm{mult}(rc)}.$$

Definition 46.3.2. $\mathrm{Cut}_0(\xi)$ is the set of classes modulo $K_0^Y(\xi)$ of the elements of K_M, with $\mathrm{mult}_0 := 1$. For $1 \le j \le m-1$, let $G_0 = R_j(\mathbb{A}_f)$ and $G_1 = R_j(\mathbb{A}_f) \cap K_j^{Y\circ}$ act on $U(Y_j)(\mathbb{A}_f)/K_j^Y(\xi)$; write mult_j for the multiplicity above. A *cut* at stage j is a finite set $\mathrm{Cut}_j(\xi)$ of classes such that: (a) it is G_1-stable; (b) its points are integral at every place not above a prime dividing $\mathrm{Nm}\, Q_\xi$; (c) on it mult_j divides a power of $\mathrm{Nm}\, Q_\xi$; (d) $\mathrm{mult}_j(x) \neq 0$ whenever the G_0-orbit of x contains a point with graded lattice and the lattice bounds at stage j. The proof fixes one cut at each stage.

For $1 \le j < m$ the proof shows that a cut exists by choosing, inside the image of $\mathcal{Z}_j K_j^{Y\circ}$ modulo $K_j^Y(\xi)$, one class in each G_0-orbit meeting that image, and taking the G_1-saturation. For x integral off $Q_\xi \cup \bar{Q}_\xi$, such as the points of $\mathcal{Z}_j K_j^{Y\circ}$, it shows that there is n with $(xux^{-1})^{(\mathrm{Nm}\, Q_\xi)^n} \in K_j^{Y\circ}$ for all $u \in K_j^Y(\xi)$.

Problem 46.3.3. Let C meet each G_0-orbit in at most one point, and $\mathrm{Cut} = G_1 \cdot C$. For $c \in C$, $\mathrm{mult}(c) = [\mathrm{Stab}_c : \mathrm{Stab}_c \cap G_1]$ with $\mathrm{Stab}_c := \mathrm{Stab}_{G_0}(c)$ (index 0 if infinite); and $\mathrm{mult}(st) = \mathrm{mult}(t)$ for $s \in G_0$.

Problem 46.3.4. (a) A finite group in which every element has order dividing q^n has order dividing a power of q. (b) For $u \equiv 1 \pmod p$ in $\mathrm{GL}_2(\mathbb{Z}_p)$ and $x = \mathrm{diag}(p^{-2}, 1)$, $xu^p x^{-1}$ is integral; for $p = 3$, $u = \left(\begin{smallmatrix} 4 & 3 \\ 3 & 1 \end{smallmatrix}\right)$, xux^{-1} is not. (c) For the G_1-saturation of the classes chosen in the construction above, every c has $\mathrm{mult}_j(c) \mid (\mathrm{Nm}\, Q_\xi)^n$ for some $n \ge 0$.

Problem 46.3.5. For $0 \le j < m$, if $Q_\xi = \varnothing$, every finite point representing a class of $\mathrm{Cut}_j(\xi)$ lies in $K_j^{Y\circ}$, and $\#\mathrm{Cut}_j(\xi) \le \#\mathrm{GL}_{r+jt}(\mathcal{O}_E/\pi_M) \le M^{f(r+jt)^2} \le M^{fn_m^2}$.

46.4. Summing over the cuts

Fix $1 \le j \le m-1$, and let G_0, G_1 be as in Definition 46.3.2. A level K' here means a subgroup of $K_j^{Y\circ}$ containing a neighbourhood of the identity.

Problem 46.4.1. Let F_∞ be a function on $U(Y_j)_\infty$. There is $k(F_\infty) \in \mathbb{C}$ with the following property, for every level K', every finite G_1-stable set Cut of classes modulo K', every complex function h_1 on $U(Y_j)(\mathbb{A}_f)/K'$ and every continuous left $R_j(F)$-invariant function f_1 on $U(Y_j)(\mathbb{A})$, integrable on $R_j(F)\backslash U(Y_j)(\mathbb{A})$, with

$\mathrm{mult}(x_f) \neq 0$ wherever $f_1(x) \neq 0$. If $\int_{[R_j]} f_1(rx)\,dr = [K_j^{Y\circ} : K']\, F_\infty(x_\infty)\, h_1(x_f K')$ whenever $x_f K' \in \mathrm{Cut}$, then

$$\int_{R_j(F)\backslash U(Y_j)(\mathbb{A})} f_1 = k(F_\infty) \sum_{c \in \mathrm{Cut}} \frac{h_1(c)}{\mathrm{mult}(c)}.$$

Problem 46.4.1 uses quotient integration in stages [Bou, Ch. VII, §2]; *Status*: classical. Problems 46.1.1 and 46.4.1, with Problem 45.1.2 and Problem 46.2.16 for the support, give the following by induction. Fix g, λ, a stage $0 \leq j \leq m-1$, and the grading of Y_{j+1} used in the stage function. For every complex function F_∞ of the tuple of archimedean outer points there is $c_j(F_\infty) \in \mathbb{C}$ with the following property. Let ξ be tame, $\varphi \in S_{\mathrm{ram}}$, z a form on Y_{j+1} as in Problem 45.1.2, and S_1 a complex function of the tuple of finite level classes. Suppose that for every tuple $(g_k, y_k)_k$ with $y_0 \in K_M$ and $y_m = 1$, the integral of $\mathcal{W}_{j+1} z$ over $R_{j+1}(F)\backslash U(Y_{j+1})(\mathbb{A})$ equals $F_\infty((g_k)_k)\, S_1((y_k K_k^Y(\xi))_k)$. Then the integral of $z\,\Theta_{\xi,j+1} \cdots \Theta_{\xi,1} \mathcal{P}_{g,\lambda}\varphi$ over $[U(Y_{j+1})]$ equals

$$c_j(F_\infty) \sum_{c_0 \in \mathrm{Cut}_0(\xi), \ldots, c_j \in \mathrm{Cut}_j(\xi)} \frac{S_1(c_0, \ldots, c_j, 1, 1, \ldots)}{\prod_{k=0}^{j} \mathrm{mult}_k(c_k)},$$

where each 1 denotes the identity class at its stage.

For interlacing highest weights, restriction from $U(n+1)$ to $U(n)$ has multiplicity one [GW, Thm. 8.1.1]; *Status*: classical. For the weights and integral Weyl modules of X_m, Y_m, the proof fixes an integer matrix whose complex span is the space of intertwiners for the block embedding.

Problem 46.4.2. Let l be a linear form on the value space of the algebraic forms on X_m of the prescribed weight. There is $c(l) \in \mathbb{C}$ with the following property, for every such form ψ and every h' with $h'(g_\infty, x) = \langle v^+, \rho(g_\infty^{-1}) P'(x) \rangle$ for an algebraic form P' on Y_m of the weight of the lower chain. Identify Y_m with the complement of the last basis vector of X_m. If $F'(k)$ is the coefficient of ψ at l, evaluated at $\mathrm{diag}(k, 1)$, for every $k \in U(Y_m)(\mathbb{A})$, then

$$\int_{[U(Y_m)]} F'\, \overline{h'} = c(l) \int_{[U(Y_m)]} (\mathrm{res}\psi)\, \overline{h'},$$

where v^+ is the highest-weight vector, ρ the action on the value space, the product is conjugate-linear in the first variable, and on the right $\mathrm{res}\psi$ is read in the same way.

Definition 46.4.3. The *seesaw tensor* of a tame character ξ is

$$\mathrm{ss}_\xi := \sum_{c_0 \in \mathrm{Cut}_0(\xi), \ldots, c_{m-1} \in \mathrm{Cut}_{m-1}(\xi)} \Big(\prod_{j=0}^{m-1} \frac{1}{\mathrm{mult}_j(c_j)} \Big) \big(\phi_\varrho^{\mathrm{ev}}(\xi, c) \otimes \cdots \otimes \phi_0^{\mathrm{ev}}(\xi, c) \big),$$

with $c_m := 1$, in the order of Chapter 45; the evaluated data do not depend on the representatives of the classes.

In the proof $\Theta_{\xi,j+1}^{\top}$ preserves the forms of Problem 45.1.2, and both sides below have the shape of h' in Problem 46.4.2, where pairing with such h' is nondegenerate. So Problems 46.1.1, 46.2.16, 46.4.1 and 46.4.2, with Theorem 45.5.1 and the facts quoted, prove the following.

Theorem 46.4.1. *Let λ be a linear form on the values of source forms and g a point of the archimedean unitary group of Y_0. There is $c' \in \mathbb{C}$, possibly zero, such that for every tame character ξ and every $\varphi \in S_{\mathrm{ram}}$*

$$\Theta_{\xi,m} \cdots \Theta_{\xi,1} \mathcal{P}_{g,\lambda} \varphi = c' \operatorname{res} \Theta_{\xi}^{X}(\mathrm{ss}_{\xi})\varphi,$$

an equality of automorphic forms on Y_m, with res *read in automorphic forms.*

In the Lean formalization. Theorem 46.4.1 is `exists_lowerChainUpTo_poincareAt_eq` in the Lean proof [GL]; the string parts Λ^-, Λ^+ are `latticeMinus`, `latticePlus`; $\mathfrak{u}$ is `twistIdele`; the lattice bounds are `HasLatticeBounds`; a split lattice is `LatticeSplitsAt`; the cuts and multiplicities are `cutPoints` and `cutMultiplicity`; and ss_{ξ} is `seesawDatum`. Problem 46.2.16 is `hasLatticeBounds_of_unfoldStage_ne_zero`. Other assertions attributed to the proof: the one-lift inclusion (in general), the standard image, the string parts of the standard lattice, fact (ii), the construction of a cut, the common conjugate power, the integer branching matrix, the transpose on conjugate-type forms and the comparison by pairing.

We have proved Theorem 46.4.1. Taken as given: the one-lift inclusion and the image of the standard lattice, the string parts of the standard lattice, facts (i) and (ii), the existence of a cut, quotient integration in stages, the branching rule with its integer matrix, and the statements marked "in the proof". c' may vanish, and ss_{ξ} is not yet a combination of translated vacua. The next step needs Definitions 46.3.2 and 46.4.3 and Problems 46.3.4 and 46.3.5.

Hints

§46.1.

46.1.1: Then the transpose relation, the definition of $\mathcal{P}_{g,\lambda}$ and Problem 45.1.2; $\Theta_{\xi,1}^{\top} z$ is $K_0^Y(\xi)$-fixed and the K_M-trace of its conjugate lies in $H(Y_0)$.

§46.2.

46.2.3: Λ is spanned by $(1, \frac{1}{5})$ and $(0, 1)$.
46.2.4: A vector of W pairs with $v \in \Lambda$ as with the projection of v.
46.2.5: Translation invariance of the function.
46.2.9: Apply Problem 46.2.4 to the upper bound at stage j.
46.2.11: Dualize the lower bound; work at a place above N (where $\mathfrak{u}$ is a unit) at which $\mathfrak{u}^j v$ is nonzero in the local quotient; the quotients at w and $\bar{w}$ are dual (Problem 46.2.4). m is even.
46.2.14: Problems 43.5.2 and 43.5.3, and the equivariance $\Theta(\omega^X(g,h)f) = h \circ \Theta(f) \circ g^{-1}$ of each lift. Note $T_w^{N-1} = 1$ on S.
46.2.15: With (g_∞, x) as outer variable of stage j, Theorem 45.5.1 and its proof give $\mathcal{W}_j$ on $R_j(\mathbb{A})(g_\infty, x)$ via the first $j/2$ double lifts.
46.2.16: Strong induction. At stage zero the point $\gamma_f y_0$ has the translate y_0, with lattice $\Lambda(1)$. Later, use (i) and the earlier bounds where the integrand is nonzero, then Problems 46.2.9, 46.2.11 and 46.2.15.

§46.3.

46.3.1: H is transitive on H/H_0.

46.3.3: The classes counted at c form a Stab_c-orbit in G_0/G_1.

46.3.4: (a) Cauchy's theorem. (b) $u^p \equiv 1 \pmod{p^2}$. (c) At a selected class c, $[\mathrm{Stab}_c : \mathrm{Stab}_c \cap G_1] < \infty$; let Stab_c act on $\mathrm{Stab}_c/(\mathrm{Stab}_c \cap G_1)$; Problem 46.3.3.

46.3.5: Reduce integral representatives modulo π_M; $\#\mathcal{O}_E/(\pi_M) = M^f$.

§46.4.

46.4.1: A cutoff $\mathfrak{k} \geq 0$ of $R_j(\mathbb{A})$-orbit integral 1; the Haar measure du on G_0 giving G_1 measure 1; weight $\int_{G_0} \mathfrak{k}(x_\infty, u)\, du$ times $1/\mathrm{mult}(x_f K')$ on the cut, 0 off it. Compare Problem 43.3.2.

46.4.2: Average $v \mapsto l(v)v^+$ over $U(Y_m)_\infty$ (a model: $\frac{1}{4}\sum_{u \in \mu_4} u^{-1}(2+3u) = 3$ extracts one weight), then use branching and Schur orthogonality; $R_m = U(Y_m)$.

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