TWO LIFTS AT A TIME

Prerequisites. Chapter 44: spaces with a string, grades and degrees, string maps, constant terms, the transfer (Definitions 44.2.1–44.4.2), the Bessel groups $R = \mathcal{U} \rtimes U(X)$, ψ_E, j_E, $\psi_{\mathcal{U}}$, and Theorem 44.4.1, in its notation. Chapter 43, Definitions 43.4.1 and 43.4.2: the Bessel function $\mathcal{W}_{\lambda,\varphi}$; lift j of the lower chain $\Theta_{\xi,j}$, its Weil representation ω_j, its datum $\phi_j(\xi)$ (a harmonic at infinity times the finite datum $\mathbf{1}_j(\xi)$), the levels $K_j^Y(\xi) \subseteq K_j^{Y\circ}$; Heisenberg translations Heis; the definite lifts $\Theta^X_{\xi,j}(v)$. Chapter 23: Problems 23.2.3, 23.4.1, 23.5.1. Chapters 33, 34: X_j, Y_j, $\varrho = m/2 - 1$, tame characters (of weight zero), Problem 33.5.2.

In the proof of the seesaw identity (Chapter 43) the Bessel period of φ is transferred along the lower chain one lift at a time (Theorem 44.4.1). How does the function left on Y_j become a lift between the *definite* spaces X_j? The proof groups the transfers in pairs: the string maps of two consecutive lifts fit into one isotropic map. We compute it for strings of lengths $3, 2, 1$; then, after each pair, we identify the translated stage function on the definite group.

Lift j ($1 \le j \le m$) goes from Y_{j-1} to Y_j; pair i ($0 \le i \le \varrho$) consists of lifts $2i+1$, $2i+2$. For $j < m$, Y_j is X_j plus a string of length $m-1-j$, with Bessel group R_j; Y_m is definite, of rank $n_m - 1$, with string of length 0 and $R_m = U(Y_m)$ (Chapter 33).

45.1. Stage functions

Fix λ, archimedean points g_j of $U(Y_j)$, finite points $y_j \in U(Y_j)(\mathbb{A}_f)$ ($j \ge 0$), a tame character ξ and $\varphi \in S$.

Definition 45.1.1. Put $\mathcal{W}_0^{\mathrm{tr}} := \mathcal{W}_{\lambda,\varphi}$ on $R_0(\mathbb{A})$. For $0 \le j \le m-1$ let

$$\mathcal{W}_{j+1}(h) := \int_{[R_j]} \mathcal{W}_j^{\mathrm{tr}}(r)\, \mathrm{CT}_{\mathrm{st}_{j+1}}\big(\omega_{j+1}(r\,(g_j, y_j), h)\, \phi_{j+1}(\xi)\big)\, dr$$

on $U(Y_{j+1})(\mathbb{A})$, the transfer of $\mathcal{W}_j^{\mathrm{tr}}$ for $\omega_{j+1}((g_j, y_j), 1)\, \phi_{j+1}(\xi)$, and, for $j+1 < m$, $\mathcal{W}_{j+1}^{\mathrm{tr}}(r) := \mathcal{W}_{j+1}(r\,(g_{j+1}, y_{j+1}))$ on $R_{j+1}(\mathbb{A})$. These are the *stage functions*.

In the proof, for $0 \le j < m$, $\mathcal{W}_j^{\mathrm{tr}}$ satisfies the hypothesis of Definition 44.4.2 (continuous, left $R_j(F)$-invariant, $\mathcal{W}_j^{\mathrm{tr}}(nr) = \psi_{\mathcal{U}}(n)\mathcal{W}_j^{\mathrm{tr}}(r)$ when $n-1$ has degree ≥ 2); call such functions *equivariant*. A *form of the conjugate type* on Y_j is defined as a form of the lower chain on Y_j, with the type kernel replaced by its complex conjugate; these forms are paired with those of the lower chain. In the proof, for $0 \le j < m$ there is a linear map $\Theta^{\top}_{\xi,j+1}$, the *transpose* of lift $j+1$, from the forms of the conjugate type on Y_{j+1} to those on Y_j, with

$$\int_{[U(Y_{j+1})]} z\ \Theta_{\xi,j+1} f = \int_{[U(Y_j)]} (\Theta^{\top}_{\xi,j+1} z)\, f$$

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for every form f of the lower chain on Y_j and every form z of the conjugate type on Y_{j+1}; and for such z,

$$(\Theta^{\top}_{\xi,j+1}z)(g) = [K_j^{Y\circ} : K_j^Y(\xi)] \int_{[U(Y_{j+1})]} \theta\big(\omega_{j+1}(g,h)\,\phi_{j+1}(\xi)\big)\, z(h)\, dh,$$

and z is automorphic and rapidly decreasing.

Problem 45.1.2. For $0 \le j \le m-1$ and every form z of the conjugate type on Y_{j+1}, the integral on the right converges absolutely and

$$\int_{[R_j]} \mathcal{W}_j^{\mathrm{tr}}(r)\,(\Theta^{\top}_{\xi,j+1}z)\big(r\,(g_j,y_j)\big)\,dr = [K_j^{Y\circ} : K_j^Y(\xi)] \int_{R_{j+1}(F)\backslash U(Y_{j+1})(\mathbb{A})} \mathcal{W}_{j+1}\, z.$$

Chapter 46 uses this identity in the recursion for paired periods.

45.2. Three strings

For adelic maps and data we use the CM field, Weil data and framed spaces of Chapter 44. Write V^- for V with its form negated.

Definition 45.2.1. Let V, V', V'' be spaces with strings for the same N, of lengths $\ell+1, \ell, \max(\ell-1, 0)$ $(\ell \ge 0)$, with string maps $\mathrm{st}_1 \colon V \to V'$ and $\mathrm{st}_2 \colon V' \to V''$. The *stacked string map* is $A := \binom{\mathrm{st}_1^*}{\mathrm{st}_2} \colon V' \to V^- \oplus V''$.

In Problems 45.2.2–45.2.4 take $E = \mathbb{Q}(\mathrm{i})$, $N = 5$, $c_V = c_{V'} = c_{V''} = 1$, no definite parts, $J_V = \mathrm{antidiag}(1,-5,1)$, $J_{V'} = \mathrm{antidiag}(1,1)$, $J_{V''} = (-5)$.

Problem 45.2.2. $\mathrm{st}_1 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \end{smallmatrix}\right)$, $\mathrm{st}_2 = (1,0)$, and $\mathrm{st}_1\mathrm{st}_1^* = \mathrm{st}_2^*\mathrm{st}_2 = \left(\begin{smallmatrix} 0 & 0 \\ -5 & 0 \end{smallmatrix}\right)$.

Problem 45.2.3. A has rows $(0,0)$, $(1,0)$, $(0,1)$, $(1,0)$, and $A^*A = 0$ for the form $-J_V \oplus J_{V''}$.

Problem 45.2.4. For the grades of V' and of V as weights, the maps $a \colon V \to V'$ of positive degree are the $a = \left(\begin{smallmatrix} 0 & \alpha_0 & \alpha_1 \\ 0 & 0 & \alpha_2 \end{smallmatrix}\right)$; then $a^* = \left(\begin{smallmatrix} \bar\alpha_2 & \bar\alpha_1 \\ 0 & -\bar\alpha_0/5 \\ 0 & 0 \end{smallmatrix}\right)$, and $a \mapsto a^*$ is a conjugate-linear bijection onto the maps $V' \to V$ of positive degree.

Problem 45.2.5. Over $\mathbb{Q}(\mathrm{i})$, for three spaces with no definite parts, string lengths $4,3,2$, $N = 5$ and $c_V = c_{V'} = c_{V''} = 2$: $\mathrm{st}_1 = \left(\begin{smallmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{smallmatrix}\right)$, $\mathrm{st}_2 = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -10 & 0 \end{smallmatrix}\right)$, and $\mathrm{st}_1\mathrm{st}_1^* = \mathrm{st}_2^*\mathrm{st}_2 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -10 & 0 \end{smallmatrix}\right)$.

Problem 45.2.6. Let $\ell \ge 1$ and $c_V = c_{V'}$. Then $\mathrm{st}_1\mathrm{st}_1^* = \mathrm{st}_2^*\mathrm{st}_2$; this map kills the definite part and $f'_{\ell-1}$ and sends f'_q to a multiple of f'_{q+1} for $q < \ell-1$. Also $A^*A = -\mathrm{st}_1\mathrm{st}_1^* + \mathrm{st}_2^*\mathrm{st}_2 = 0$.

In pair i the strings of $Y_{2i}, Y_{2i+1}, Y_{2i+2}$ have lengths $2(\varrho-i)+1$, $2(\varrho-i)$, $2(\varrho-i)-1$ if $i < \varrho$ (a *middle pair*) and $1,0,0$ if $i = \varrho$ (the *last pair*). The gradings of Y_j, $j < m$, have one common constant; that of Y_m has constant 1. In a middle pair the first string has odd length, so a map $V' \to V^- \oplus V''$ homogeneous of degree zero kills the string of V' and maps its definite part into the span of the definite parts and the middle vectors f_{mid} of V and f'_{mid} of V'', of norms N in V^- and $-N$ in V''.

Problem 45.2.7. For norms 5, -5, the basis $\frac{1}{10}(f_{\mathrm{mid}} - f'_{\mathrm{mid}})$, $f_{\mathrm{mid}} + f'_{\mathrm{mid}}$ of the plane they span has Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; in general so has $(f_{\mathrm{mid}} - f'_{\mathrm{mid}})/2N$, $f_{\mathrm{mid}} + f'_{\mathrm{mid}}$.

So in a middle pair the part of grade zero of $Y_{2i}^- \oplus Y_{2i+2}$ is $X_{2i}^- \oplus X_{2i+2}$ plus one hyperbolic plane.

Problem 45.2.8. In the last pair, with f_{mid} the vector of the string of $Y_{2\varrho}$, f_{mid}/π_N has norm 1 in $Y_{2\varrho}^-$, for $N = 5$, $\pi_N = 2 + \mathrm{i}$ and in general. The part of grade zero of $Y_{2\varrho}^- \oplus Y_m$ is $X_{2\varrho}^- \oplus X_m$, no plane being left, with $X_m = Y_m \perp E\, f_{\mathrm{mid}}/\pi_N$, and $h \in U(Y_m)$ acts on X_m as $\mathrm{diag}(h, 1)$.

Problem 45.2.9. In the last pair $A = 0$.

In the proof, $\mathrm{Heis}(w)\mathrm{Heis}(w') = \psi_E\big(\frac{1}{2} j_E \,\mathrm{tr}(w'^* w)\big)\mathrm{Heis}(w + w')$ for adelic maps w, w' between two spaces, the adjoint taken for their forms; we use it for $\mathrm{Hom}(V', V^- \oplus V'')$.

Problem 45.2.10. Under the hypotheses of Problem 45.2.6, $(An')^*(An) = 0$ and $\mathrm{Heis}(An)\mathrm{Heis}(An') = \mathrm{Heis}(A(n + n'))$ for all adelic endomorphisms n, n' of V'.

45.3. Two constant terms at once

Let V, V', V'' be spaces with strings for the same N, of lengths $\ell + 1, \ell, \max(\ell - 1, 0)$, $\ell \geq 0$ even, $c_V = c_{V'}$. For the splitting into definite spaces and the two-period identity take $V = Y_{2i}$, $V' = Y_{2i+1}$, $V'' = Y_{2i+2}$ with the middle or last double-lift datum of pair i. Write X, X', X_0'' for the definite parts and R, R' for the Bessel groups of V, V'. For data ϕ on $\mathrm{Hom}(V, V')$ and ϕ' on $\mathrm{Hom}(V', V'')$, the *join* $\phi \boxtimes \phi'$ on $\mathrm{Hom}(V', V^- \oplus V'')$ is, in the proof, the product of ϕ' with ϕ carried to $\mathrm{Hom}(V', V^-)$ by a scaled adjoint change of coordinates; for all adelic w, w', with matrices in the rational bases, $\mathrm{Heis}\binom{w^*}{w'}(\phi \boxtimes \phi') = \mathrm{Heis}(w)\phi \boxtimes \mathrm{Heis}(w')\phi'$. We use:

(i) *Product formula*, in the proof: for framed spaces V, V', V'' (definite ones included) there is $z_\theta \neq 0$ with $\theta(\phi \boxtimes \phi') = z_\theta\, \theta(\phi)\theta(\phi')$ for all ϕ, ϕ'. Compare Problem 23.4.1; for the Weil representation see [We].

For data on $\mathrm{Hom}(V', V^- \oplus V'')$ let $\mathrm{zt}_{>0}(\cdot)(0)$ be the mean of $\theta(\mathrm{Heis}(b)\,\cdot\,)$ over the adelic maps $b\colon V' \to V^- \oplus V''$ of positive degree modulo the rational ones (the weights on $V^- \oplus V''$ are the grades); at a point v of the retained grade-zero space, $\mathrm{zt}_{>0}(\phi)(v)$ is the same mean of $\theta(\mathrm{Heis}(\iota v + b)\phi)$, ι the inclusion of that space.

Problem 45.3.1. For adelic maps $w\colon V \to V'$ and $w'\colon V' \to V''$,

$$\mathrm{CT}_w\phi \cdot \mathrm{CT}_{w'}\phi' = z_\theta^{-1}\,\mathrm{zt}_{>0}\big(\mathrm{Heis}(w)\phi \boxtimes \mathrm{Heis}(w')\phi'\big)(0).$$

Problem 45.3.2. If $\mathrm{len}\, V$ is odd, all grades of V are even, and $n - 1$ has degree ≥ 2 for every $n \in \mathcal{U}(\mathbb{A})$.

In the proof, for the first edge with the Weil action ω of the rank twists of an automorphic character of the norm-one torus, $\mathcal{W}(nr)\,\mathrm{CT}_{\mathrm{st}_1}(\omega(nr, h)\phi) = \mathcal{W}(r)\,\mathrm{CT}_{\mathrm{st}_1}(\omega(r, h)\phi)$ for every equivariant $\mathcal{W}$, $n \in \mathcal{U}(\mathbb{A})$, $r \in R(\mathbb{A})$, $h \in U(V')(\mathbb{A})$. Let zt be the zero term along the maps of positive degree followed, in a middle pair, by the mean over the adelic maps modulo the rational maps from X' into the line $E(f_{\mathrm{mid}} + f'_{\mathrm{mid}})$ of Problem 45.2.7 (no plane in the last pair). It lands in data on $\mathrm{Hom}(X', X^- \oplus X'')$, where $X'' = X_0''$ in a middle pair and $X'' = X_m$ in the last pair.

(iv) *Bessel groups*, in the proof. Let $R = \mathcal{U} \rtimes U(X)$ be the Bessel group of a space W with a string; $[R]$ is compact. Let $b_1, \ldots, b_e$ be distinct basis vectors of grade one ($e \geq 0$; none if $\mathrm{len}\, W$ is odd) and $n^{(1)} := (c_W(n - 1)_{b_p, x})_{p,x}$, x running over

the basis of X. There is c_R such that: if H is continuous and left $R(F)$-invariant on $R(\mathbb{A})$, $H(n\gamma) = G_\gamma(n^{(1)})$ for $n \in \mathcal{U}(\mathbb{A})$, $\gamma \in U(X)(\mathbb{A})$, each G_γ is continuous and periodic under rational matrices, and its mean $\langle G_\gamma \rangle$ is continuous and left $U(X)(F)$-invariant, then $\int_{[R]} H = c_R \int_{[U(X)]} \langle G_\gamma \rangle \, d\gamma$. See [Bo].

For a middle or last pair we use the following two-period identity from the proof. Let ω, ω' be the Weil representations of the two edges, with automorphic characters μ, μ' of the norm-one torus, each multiplied on a space of rank d by $v^{\lfloor d/2 \rfloor}$. For data ϕ, ϕ', an equivariant $\mathcal{W}$ on $R(\mathbb{A})$ and $h \in U(X_0'')(\mathbb{A})$, acting on V'' through X_0'',

$$\int_{[R']} \int_{[R]} \mathcal{W}(r) \, \mathrm{CT}_{\mathrm{st}_1}\big(\omega(r, r')\phi\big) \, \mathrm{CT}_{\mathrm{st}_2}\big(\omega'(r', h)\phi'\big) \, dr \, dr'$$
$$= \frac{c_R c_{R'}}{z_\theta} \int_{[U(X)]} \mathcal{W}(\gamma) \int_{[U(X')]} \theta\big(\omega^{\triangle}(\gamma, \gamma', \bar{h}) \, \mathrm{zt}\,\mathrm{Heis}(A)(\phi \boxtimes \phi')\big) \, d\gamma' \, d\gamma, \quad (1)$$

where $c_R, c_{R'}$ do not depend on $\mu, \mu', \phi, \phi', \mathcal{W}$ or h; $\omega^{\triangle}$ is the Weil representation of $U(X) \times U(X') \times U(X'')$ on $\mathrm{Hom}(X', X^- \oplus X'')$ normalized as in the proof (the characters μ, μ' and a flip character); and $\bar{h} = h$ in a middle pair, $\bar{h} = \mathrm{diag}(h, 1)$ in the last.

45.4. The definite double lift

The right side of (1) is a double theta integral between definite spaces. First the Gaussian case.

Problem 45.4.1. By machine, $c_4 = 0.143191337023821\ldots$, and the composite $\mathcal{A}_{(4)}(1) \to \mathcal{A}_{(4)}(1) \to \mathcal{A}_{(4)}(1)$ of two lifts $\Theta_{(4)}$ is multiplication by $c_4^2/16 = 0.00128148493741\ldots$.

In a model with several classes a form is a vector indexed by classes, a lift has a kernel matrix K, the class weights form a diagonal matrix D, the lift is $v \mapsto KDv$, and $\langle z, v \rangle := z^t D v$.

Problem 45.4.2. For $K_0 = \left(\begin{smallmatrix} 1 & 2 \\ 3 & -1 \end{smallmatrix}\right)$, $K_1 = \left(\begin{smallmatrix} 2 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, $D_0 = \mathrm{diag}(1, \frac{1}{2})$, $D_1 = \mathrm{diag}(\frac{1}{3}, 1)$, $D_2 = \mathrm{diag}(1, \frac{1}{4})$, $v = (2, 1)^t$, $z = (1, -2)^t$: $K_1 D_1 K_0 D_0 v = (\frac{15}{2}, -\frac{11}{2})^t$, $z^t D_2 K_1 D_1 K_0 D_0 v = \frac{41}{4}$, and with both lifts transposed onto z, $K_0^t D_1 K_1^t D_2 z = (\frac{31}{6}, -\frac{1}{6})^t$, whose pairing $(\cdot)^t D_0 v$ with v is again $\frac{41}{4}$.

Each datum $\phi_j(\xi)$ is a pure tensor $\mathbf{u} \otimes \phi_f$ of a real Fock vector $\mathbf{u}$ and a finite datum ϕ_f (the integrated Heisenberg action of the proof). In the proof: the Weil actions, the join and $\mathrm{Heis}(w)$ respect this separation, a translate staying pure with any scalar absorbed in its real vector; on the real factor the characters act through their weights, which do not depend on the tame character; and the finite join $\boxtimes$ on tensors of finite data is injective, with a linear left inverse $\boxtimes^{-1}$.

(ii) *Archimedean zero term*, in the proof: for a translate $\mathrm{Heis}(w)$ and a real Fock vector $\mathbf{u}$ there is a real Fock vector $\mathbf{u}_0$ with $\mathrm{zt}(\mathrm{Heis}(w)(\mathbf{u} \otimes \phi_f)) = \mathbf{u}_0 \otimes \mathrm{zt}_f(\mathrm{Heis}_f(w_f)\phi_f)$ for every ϕ_f; here zt_f is the finite zero term and w_f the finite coordinate of the translation in the Weil model (the finite part of w, conjugated into the frames and divided by the finite idele of the Weil data). For the mixed model see [MVW], [We].

Definition 45.4.3. For $0 \le i \le \varrho$, with A and zt those of pair i, the *evaluated pair datum* is

$$\phi_i^{\mathrm{ev}}(\xi, y) := \boxtimes^{-1} \mathrm{zt}_f \,\mathrm{Heis}_f(A_f)\big(\mathbf{1}_{2i+1}(\xi)^y \boxtimes \mathbf{1}_{2i+2}(\xi)^y\big),$$
$$\text{where } \mathbf{1}_j(\xi)^y := \omega_j(y_{j-1}, y_j)\,\mathbf{1}_j(\xi),$$

with A_f as in (ii); it is a tensor of finite data on $\mathrm{Hom}(X_{2i}, X_{2i+1})$ and on $\mathrm{Hom}(X_{2i+1}, X_{2i+2})$.

In the proof, $\boxtimes\phi_i^{\mathrm{ev}}(\xi, y)$ is the finite zero term itself. Put $\phi_j(\xi)^{g,y} := \omega_j((g_{j-1}, y_{j-1}), (g_j, y_j))\,\phi_j(\xi)$.

Problem 45.4.4. For the fixed pair data and harmonics there is a real Fock vector $\mathbf{u}_i$, depending on the outer points only through $g_{2i}, g_{2i+1}, g_{2i+2}$, with $\mathrm{zt}\,\mathrm{Heis}(A)(\phi_{2i+1}(\xi)^{g,y} \boxtimes \phi_{2i+2}(\xi)^{g,y}) = \mathbf{u}_i \otimes \boxtimes\,\phi_i^{\mathrm{ev}}(\xi, y)$ for every tame ξ and all finite points y.

(iii) *Definite theta lifts*, in the proof. For a form ψ of the prescribed weight on X and a linear form l on its values put $\mathrm{coeff}_l\,\psi(\gamma) := l(\gamma_\infty^{-1}\psi(\gamma_f))$, γ_∞ acting on the values. For a family of definite lifts Θ_o^X, $o \in O$, between fixed spaces of algebraic forms of the prescribed source and target weights, with Weil actions ω_o, the coefficient property is: for every polynomial P on $\mathrm{Hom}(X, X')_\infty$, with Fock vector v_P, and every l there is l' such that for all o, finite data ϕ_f, forms ψ and $h \in U(X')(\mathbb{A})$,

$$\int_{[U(X)]} \theta\big(\omega_o(\gamma, h)(v_P \otimes \phi_f)\big)\ \mathrm{coeff}_l\,\psi(\gamma)\,d\gamma = \mathrm{coeff}_{l'}\big(\Theta_o^X(\phi_f)\psi\big)(h).$$

In the proof the lifts of the definite chain, O the tame characters, have it. See [Gr], [KV], [Ho].

Problem 45.4.5. Let $\Lambda\colon \mathsf{L} \to \mathsf{V}$ be $\mathbb{C}$-linear from a finite-dimensional complex vector space to a Hausdorff complex topological vector space, and $I\colon \mathsf{H} \to \mathsf{V}$ continuous on a topological space. If $I(u) \in \Lambda(\mathsf{L})$ for u in a dense subset of H, then for all $u \in \mathsf{H}$.

For polynomials P on $\mathrm{Hom}(X, X')_\infty$ and Q on $\mathrm{Hom}(X', X'')_\infty$, let PQ be their product after transposing the variables of P into the X-block and placing those of Q in the X''-block. In the proof, for finite data v, v' on the two edges, $(v_P \otimes v) \boxtimes (v_Q \otimes v') = v_{PQ} \otimes (v \boxtimes v')$, and on such joins the triple action at (γ, γ', h) acts on the two factors by the rank-twisted Weil actions of $X \to X'$ at (γ, γ') and of $X' \to X''$ at (γ', h).

Problem 45.4.6. Fix automorphic characters μ_o, μ_o' of the norm-one torus, $o \in O$, and families $\Theta_{1,o}^X\colon X \to X'$, $\Theta_{2,o}^X\colon X' \to X''$ with the same intermediate space of forms, having (iii) for the rank twists of μ_o and μ_o'; let $\omega_o^\triangle$ be their triple action. For P, Q as above and every l there is l'' such that for all o, finite data ϕ_f, ϕ_f', forms ψ on X and $h \in U(X'')(\mathbb{A})$,

$$\int_{[U(X)]} \mathrm{coeff}_l\,\psi(\gamma) \int_{[U(X')]} \theta\big(\omega_o^\triangle(\gamma, \gamma', h)(v_{PQ} \otimes (\phi_f \boxtimes \phi_f'))\big)\,d\gamma'\,d\gamma$$
$$= \mathrm{coeff}_{l''}\big(\Theta_{2,o}^X(\phi_f')\,\Theta_{1,o}^X(\phi_f)\,\psi\big)(h).$$

In the proof the polynomial vectors are dense in the real Fock model in its L^1 norm (for the Hilbert space see [Ba] and Problem 12.1.2); the families of double integrals, with pointwise topology in the other arguments, depend continuously on $\mathbf{u}$; and for fixed l the families $\mathrm{coeff}_{l''}(\cdots)$ form a finite-dimensional space as l'' varies. Every polynomial is a sum of products PQ, and both sides are linear in the finite tensor. For these same two families the proof gives the following coefficient formula: for every real Fock vector $\mathbf{u}$ and every l there is l'' with

$$\int_{[U(X)]} \mathrm{coeff}_l\,\psi(\gamma) \int_{[U(X')]} \theta\big(\omega_o^{\triangle}(\gamma,\gamma',h)(\mathbf{u}\otimes\boxtimes\Phi)\big)\,d\gamma'\,d\gamma = \mathrm{coeff}_{l''}\big(\Theta_o^X(\Phi)\psi\big)(h)$$

for all o, finite pair tensors Φ, forms ψ and h, where $\Theta_o^X(v\otimes v') := \Theta_{2,o}^X(v')\Theta_{1,o}^X(v)$, extended linearly. In the chain, l'' is chosen before the tame character. In the proof, for pair i the characters μ, μ' of (1) are those of lifts $2i+1$ and $2i+2$, the families are $\Theta_{1,\xi}^X = \Theta_{\xi,2i+1}^X$ and $\Theta_{2,\xi}^X = \Theta_{\xi,2i+2}^X$ with the rank twists of these characters, O the tame characters, and $\omega^{\triangle}$ of (1) is their triple action.

45.5. Induction on pairs

For $0 \le j < m$, a form ψ on X_j and a linear form l, let $\mathcal{W}_{l,\psi}$ on $R_j(\mathbb{A})$ be formed as $\mathcal{W}_{\lambda,\varphi}$ in Section 43.4: $\mathcal{W}_{l,\psi}(n\gamma) = \psi_{\mathcal{U}}(n)\,\mathrm{coeff}_l\,\psi(\gamma)$ for $n\in\mathcal{U}(\mathbb{A})$, $\gamma\in U(X_j)(\mathbb{A})$, with $\psi_{\mathcal{U}}(n) := \psi_E\big(\frac{1}{2}j_E\,\mathrm{tr}(\mathrm{st}_{j+1}^*\mathrm{st}_{j+1}n)\big)$ for every $n\in\mathcal{U}(\mathbb{A})$.

Problem 45.5.1. Let $\mathrm{len}\,Y_j$ be odd and $\mathcal{W}$ be equivariant on $R_j(\mathbb{A})$ with $\mathcal{W}(\gamma) = \mathrm{coeff}_l\,\psi(\gamma)$ for all $\gamma\in U(X_j)(\mathbb{A})$. Then $\mathcal{W} = \mathcal{W}_{l,\psi}$.

For finite pair tensors $\Phi_0,\dots,\Phi_i$, $\Theta_\xi^X(\Phi_i\otimes\cdots\otimes\Phi_0)$ is the composite of the first $i+1$ double lifts of the definite chain, pair j acting through $v\otimes v'\mapsto\Theta_{\xi,2j+2}^X(v')\circ\Theta_{\xi,2j+1}^X(v)$.

Theorem 45.5.1 (two lifts at a time)**.** *Fix λ, archimedean points g_k of $U(Y_k)$ $(k\ge 0)$ and $0\le i\le\varrho$. There is a linear form l on the values of forms on X_{2i+2}, independent of the tame character, of φ and of the finite points, such that for every tame character ξ, every $\varphi\in S$, all finite points $y_k\in U(Y_k)(\mathbb{A}_f)$, and every h in $U(X_{2i+2})(\mathbb{A})$ if $i<\varrho$, in $U(Y_m)(\mathbb{A})$ if $i=\varrho$,*

$$\mathcal{W}_{2i+2}\big(h\,(g_{2i+2},y_{2i+2})\big) = \mathrm{coeff}_l\Big(\Theta_\xi^X\big(\phi_i^{\mathrm{ev}}(\xi,y)\otimes\cdots\otimes\phi_0^{\mathrm{ev}}(\xi,y)\big)\varphi\Big)(\bar h).$$

Here h acts on Y_{2i+2} through its definite part, and $\bar h = h$ if $i<\varrho$, $\bar h = \mathrm{diag}(h,1)\in U(X_m)(\mathbb{A})$ if $i=\varrho$.

What has been proved. On the definite group the translated stage function after each pair is a coefficient of the definite chain, with one linear form for all tame characters, source forms and finite points. The problems prove: the isotropy of the stacked map (Problems 45.2.6 and 45.2.10), the product of two constant terms (Problem 45.3.1), the separation of the archimedean variables (Problem 45.4.4), the coefficient formula for polynomial vectors with its extension by density (Problems 45.4.6 and 45.4.5), and that an equivariant function on an odd string is determined by its values on the definite group (Problem 45.5.1). The hints indicate how Theorem 45.5.1 follows from these. Taken as given: the product formula (i), the archimedean zero term (ii), the coefficient property of the definite lifts (iii), the integration over a Bessel group (iv), the two-period identity (1) and the coefficient formula for all real Fock vectors (both stated from the proof; the problems give

their ingredients), and the statements marked "in the proof". Chapter 46 needs Definitions 45.1.1 and 45.4.3, Problem 45.1.2 and Theorem 45.5.1: for ramified source forms it applies support bounds and quotient integration to periods whose dependence on the archimedean points and the finite level classes separates.

In the Lean formalization. Theorem 45.5.1 specializes `Data.exists_unfoldStage_eq_algFormCoeff` [GL], which allows any double-lift data with the prescribed finite evaluation for pair i and an independent final archimedean point; here they are the middle or last pair and g_{2i+2}. The stage functions are `unfoldStageOn` and `unfoldStage`, the stacked map `stackedStringMap`, the evaluated pair datum `evaluatedDatum`. Identity (1) specializes `DoubleLiftData.exists_integral_constTermAtStringMap_mul_eq` (the action is `tripleWeil`); the display before Section 45.5 is `integral_theta_tripleWeil_eq_lift_comp`. Input (i) is `theta_arrowProduct`, (ii) specializes `exists_zeroTerm_rho_integrated_eq`, (iii) is `IsThetaLift`, proved for the chain by `isThetaLift_thetaKernel`, and (iv) is `isCocompact_ratBessel` with `integral_besselMeasure_eq_of_mean`. The other statements marked "in the proof" are:
`exists_unfoldStageOn_even_eq_sourceFn` and `formAdjoint_stackedStringMap_mul_stackedStringMap` (pairs, isotropy); `isEquivariant_unfoldStageOn` (equivariance of the stages); `transposedLiftY_apply`, `integral_mul_liftY` and `isRapid_conjFormY` (the transpose); `Heisenberg.Rep.mul` and `homBicharacter_eq` (the Heisenberg law); `arrowProduct` and `homHeisenbergOp_arrowProduct` (the join); `mul_constTermAtStringMap_unipotent_mul` (the first edge); `datumY`, `pairWeil_integrated_all`, `boxProd_integrated`, `CoordChange.map_integrated`, `exists_rho_integrated_eq`, `hasWeight_liftChar`, `injective_arrowTensor` and `leftInverse_arrowTensor` (the separation); `DoubleLiftData.arrowTensor_finEval` (the evaluated datum); `arrowProduct_integrated_polyVector` and `arrowProduct_twistedPairWeil` (polynomial joins); `Fock.denseRange_polyVector` (density), the remaining steps of the coefficient formula being in the proof of `integral_theta_tripleWeil_eq_lift_comp`; and `exists_unfoldStage_eq_algFormCoeff_of_sourceFn` (the order of l'' and ξ in the chain). Automorphy at stage $j+1$ is part of `ConjFormY`. In Problem 45.1.2 absolute convergence is `integrable_unfoldStage_mul` and the equality is `integral_unfoldStageOn_mul_transposedLiftY`.

Hints

§45.1.

45.1.2: Move (g_j, y_j) into the datum and apply Theorem 44.4.1 to $\mathcal{W}_j^{\mathrm{tr}}$.

§45.2.

45.2.3: The adjoint of $\binom{P}{Q}$ into $V^- \oplus V''$ is $(-P^*, Q^*)$.
45.2.4: The forms pair grades g and $-g$.
45.2.6: Compute both products on each f'_q, using $\bar{c}_V = c_V$.
45.2.8: Problem 33.5.2.

§45.3.

45.3.1: As in Problem 45.2.4, $a \mapsto a^*$ preserves positive degree; use (i) and Fubini on the compact groups.

§45.4.

45.4.1: Problem 23.2.3; keep track of the class weight in each lift; in Python, sum $\alpha^4 e^{-\pi|\alpha|^2}$ over $|\Re\alpha|, |\Im\alpha| \le 8$.

45.4.4: Split data and points into real and finite factors; apply (ii).

45.4.5: Finite-dimensional subspaces of a Hausdorff topological vector space are closed.

45.4.6: Apply (i) to the definite triple X, X', X'', with its own scalar z_θ^X; after Fubini use (iii) over γ, then over γ' for $z_\theta^X Q$.

§45.5.

45.5.1: Problem 45.3.2. For Theorem 45.5.1: apply the coefficient formula with $\mathbf{u} = (c_R c_{R'}/z_\theta)\mathbf{u}_i$, and Problem 45.5.1 at the intermediate even stages.

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