TRANSFERRING A BESSEL PERIOD THROUGH A THETA LIFT

Prerequisites. Hermitian forms over fields; Witt's extension theorem [Sch]. From Chapter 36: the Bessel subgroup $R = \mathcal{U} \rtimes U(X)$ and an example of its character (Section 36.1, Problems 36.1.1 and 36.1.2), and the finite unfolding of Problem 36.2.1. From Chapter 10: the unfolding of Problem 10.1.1. From Chapter 23: Problem 23.1.5 and Section 23.4. From Chapter 43: Theorem 43.4.1 and Problem 43.2.4. The Weil representation of a unitary dual pair and the Heisenberg representation under it, at the level of their definitions [We], [Ku, §1]. Matrices act on columns; the matrix of a map $V \to V'$ has its rows indexed by the basis of V'. In this sheet b, d, s are local letters (not the scale, the prime and the tower exponent).

The proof of Theorem 43.4.1 pairs both sides with a form on the last space of the lower chain and moves the lifts onto that form by transposition, as in Problem 43.2.4; what is left is a Bessel period of a theta function, integrated against the form. How does such a period pass through one theta lift? For the weights 0, the theta function expands in constant terms indexed by the classes of rational maps $V \to V'$ modulo maps of degree ≥ 0 (Section 44.2). In the period against a Bessel function only the $U(V')(F)$-translates of the set of classes agreeing with the string map contribute; the rational Bessel subgroup of the second space preserves this set, and each translate contributes one value of the transferred function. We first see the mechanism over $\mathbb{F}_3$ and $\mathbb{F}_9$.

44.1. Theta against a character on a plane over $\mathbb{F}_3$

Let $Q(v) = v_1^2 + v_2^2$ on $\mathbb{F}_3^2$, let $\zeta_3 := e^{2\pi \mathrm{i}/3}$, and for a function $\phi\colon \mathbb{F}_3^2 \to \mathbb{C}$ and $b \in \mathbb{F}_3$ put $(n(b)\phi)(v) := \zeta_3^{bQ(v)}\phi(v)$ and $\theta(\phi) := \sum_v \phi(v)$. The operators $n(b)$ commute with $(h\phi)(v) := \phi(h^{-1}v)$ for h in the orthogonal group $O(Q)$. In this section $\phi(v) := 1 + [v_1] + 2[v_2]$, with $[\,\cdot\,] \in \{0,1,2\}$ the least residue.

Problem 44.1.1. The sums $\sum_{Q(v)=1+j} \phi(v)$ are 13, 22, 1 for $j = 0, 1, 2$.

Problem 44.1.2. For every ϕ, $\frac{1}{3}\sum_{b\in\mathbb{F}_3} \zeta_3^{-b}\,\theta(n(b)\phi) = \sum_{Q(v)=1} \phi(v)$. For the ϕ above it is 13.

The left side is a period of $b \mapsto \theta(n(b)\phi)$ against a character; only the v with $Q(v) = 1$ survive it.

Problem 44.1.3. $O(Q)$ has 8 elements, the stabilizer of $v_0 := (1, 0)$ has 2, and the orbit of v_0 is $\{(\pm 1, 0), (0, \pm 1)\} = \{Q = 1\}$.

Problem 44.1.4. Let $\mathrm{cf}(\phi)$ be the coefficient of Problem 44.1.2 and Stab the stabilizer of v_0. For $h \in O(Q)$, $\mathrm{cf}(h\phi) = \sum_{q \in O(Q)/\,\mathrm{Stab}} \phi(h^{-1}qv_0)$; for $h = 1$ this is

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$\frac12\sum_{\gamma\in O(Q)}\phi(\gamma v_0)=13$. For every function z on $O(Q)$,

$$\sum_{h\in O(Q)} z(h)\,\mathrm{cf}(h\phi)=\sum_{h\in O(Q)}\Big(\frac{1}{|\mathrm{Stab}|}\sum_{\gamma\in O(Q)} z(\gamma h)\Big)\phi(h^{-1}v_0).$$

This period is the finite model of the transfer: a sum over $O(Q)/\operatorname{Stab}$ of one function $h\mapsto\phi(h^{-1}v_0)$, evaluated at $q^{-1}h$. In the adelic period the unipotent group can be nonabelian, and the proof uses two choices of weights.

44.2. Strings, grades and the string map

For the matrix definitions and the finite examples, let E be a field with an involution $x\mapsto\bar x$ and fixed field F, and let $N\in F^\times$; all bases are finite.

Definition 44.2.1. A *space with a string* for N is a hermitian space V over E with a basis consisting of an orthonormal basis of a subspace X and vectors $f_0,\dots,f_{s-1}$ orthogonal to X, and with a constant $c_V\in F^\times$, such that f_i pairs only with f_{s-1-i}, to 1, or to $-N$ if $i=s-1-i$. We write J_V for the Gram matrix. X is the *definite part*, $f_0,\dots,f_{s-1}$ is the *string*, and $\operatorname{len}V:=s$ is its *length*.

Definition 44.2.2. The *grade* of a vector of the basis of X is 0, and that of f_i is $s-1-2i$. Let V' be a second space with a string, and let d be integer weights on the basis of V'. For basis vectors e of V and e' of V', the entry of a map $w\colon V\to V'$ in the row of e' and the column of e has *degree* $d(e')-\operatorname{grade}(e)$, and w has degree $\ge i$ if all its entries of degree $<i$ vanish. For maps $V\to V$ the weights are the grades.

For the shape of Section 36.1 the string is $v_1,\dots,v_\varrho,z,v'_\varrho,\dots,v'_1$; $\mathcal U$ is the group of unitary n with $n-1$ raising grades, and R that of the unitary g with gf_i-f_i in the span of the string vectors of larger grade for every f_i of grade ≥0. In the proof these descriptions define $\mathcal U$ and R for every space with a string.

Definition 44.2.3. Let $\operatorname{len}V'=\operatorname{len}V-1$, or $\operatorname{len}V=\operatorname{len}V'=0$. The *string map* $\mathrm{st}\colon V\to V'$ kills the definite part and sends f_i to f'_i times a scalar, and to 0 if V' has no vector f'_i. If $\operatorname{len}V$ is odd, the scalar is 1, $-Nc_V$, c_V for f_i of positive, zero, negative grade; if $\operatorname{len}V$ is even, it is c_V, 1 for positive, negative grade. For a map $w\colon V\to V'$ put $w^*:=J_V^{-1}\bar w^tJ_{V'}$.

For the grades of V and V' the string map has degree ≥-1; in the chain of the data it is $\mathrm{st}_j\colon Y_{j-1}\to Y_j$.

The hook. In the finite examples of this section and the next, $E=\mathbb F_9=\mathbb F_3[\mathrm i]$ with $\mathrm i^2=-1$ and $\bar{\mathrm i}=-\mathrm i$, $N=1$, and V, V' have no definite part, strings of lengths 3 and 2, and constants $c_V=c_{V'}=1$: $J_V=\operatorname{antidiag}(1,-1,1)$, $J_{V'}=\operatorname{antidiag}(1,1)$, the grades are $2,0,-2$ on V and $1,-1$ on V', and $\mathrm{st}=\left(\begin{smallmatrix}1&0&0\\0&-1&0\end{smallmatrix}\right)$.

Problem 44.2.1. The unitary upper unitriangular matrices for J_V are the 27 matrices $u=\left(\begin{smallmatrix}1&a&b\\0&1&\bar a\\0&0&1\end{smallmatrix}\right)$ with $b+\bar b=a\bar a$. They form $\mathcal U$, and $u-1$ has degree ≥2 for each of them.

Problem 44.2.2. $\operatorname{tr}(\mathrm{st}^*\mathrm{st}\,u)=a-\bar a$, and for $w=(v,0,0)$ with first column $v\in\mathbb F_9^2$, $\operatorname{tr}(w^*wu)=(\bar v^tJ_{V'}v)\,b$.

44.3. The survival test

Over the adeles, let E be a CM field with the CM type and Weil data of [GL], F its maximal totally real subfield and $\mathbb{A}_E$ its adeles; ψ_E is the standard additive character of $\mathbb{A}_E/E$, and $j_E \in E^\times$, with $\bar{j}_E = -j_E$, is the element given by the Weil data. In the proof the character used on the unitary n with $n-1$ of degree ≥ 2 is $\psi_{\mathcal{U}}(n) = \psi_E\big(\frac{1}{2} j_E \operatorname{tr}(\mathrm{st}^*\mathrm{st}\, n)\big)$; the matrix $\mathrm{st}^*\mathrm{st}$ is the shift of the string of V and depends on c_V, and $\psi_{\mathcal{U}}(nn') = \psi_{\mathcal{U}}(n)\psi_{\mathcal{U}}(n')$ for two such n, n'. Over $\mathbb{F}_9$ we use the same formulas with $\psi_E(x) = \zeta_3^{\operatorname{Tr}_{\mathbb{F}_9/\mathbb{F}_3} x}$ and $j_E = \mathrm{i}$ (and $\frac{1}{2} = -1$ in $\mathbb{F}_3$), and u runs over $\mathcal{U}(\mathbb{F}_3)$.

Definition 44.3.1. Let d be weights on the basis of V'. A rational map $w\colon V \to V'$ *survives* for d if

$$\psi_E\big(\tfrac{1}{2} j_E \operatorname{tr}(\mathrm{st}^*\mathrm{st}\, u)\big) = \psi_E\big(\tfrac{1}{2} j_E \operatorname{tr}(w^* w\, u)\big)$$

for every $u \in \mathcal{U}(\mathbb{A}_F)$ such that $u-1$ has degree ≥ 2 and $w(u-1)$ has degree ≥ 1 for the weights d. A map w *agrees with* st if w and st have the same entry in the row of e' and the column of e whenever $\operatorname{grade}(e) > 0$ and $\operatorname{grade}(e') < \operatorname{grade}(e)$.

The test is applied to classes of rational maps modulo the maps of degree ≥ 0, through a chosen representative of each class; a class is determined by the entries of negative degree. For the weights 0 these are the columns of positive grade.

Problem 44.3.1. In the hook, take the weights $(0,0)$. The classes are represented by the 81 maps $w = (v, 0, 0)$, and exactly 32 of them survive: the $v \neq 0$ with $\bar{v}^t J_{V'} v = 0$.

Problem 44.3.2. The unitary group $U(V')(\mathbb{F}_3) = \{g \in \mathrm{GL}_2(\mathbb{F}_9) : \bar{g}^t J_{V'} g = J_{V'}\}$ has 96 elements, and their first columns are the 32 surviving vectors of Problem 44.3.1. The stabilizer of $(1,0)^t$ has order 3 and is the Bessel subgroup R' of V'.

Problem 44.3.3. Take the weights $(1,-1)$, the grades of V'. The classes that agree with st are represented by the nine maps $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & c & 0 \end{smallmatrix}\right)$, $c \in \mathbb{F}_9$, and exactly $c = -1$ survives: the class of st.

The hook illustrates the two choices of weights in the proof. For the weights 0 the survivors are the translates by $U(V')(\mathbb{F}_3)$ of the one class that agrees with st, $v = (1,0)^t$, in bijection with $U(V')(\mathbb{F}_3)/R'$ ($96/3 = 32$); for the grades of V' only the class of st survives among the agreeing classes. Here is a finite model of how a failing map is killed.

Problem 44.3.4. Let H be a finite group and $\mathfrak{f}\colon H \to \mathbb{C}$ a function with $\mathfrak{f}(ux) = \zeta\mathfrak{f}(x)$ for all x, for one $u \in H$ and one $\zeta \neq 1$. Then $\sum_{x \in H} \mathfrak{f}(x) = 0$. If $\mathfrak{f}$ is also left invariant under a subgroup S, then $\sum_{x \in S\backslash H} \mathfrak{f}(x) = 0$.

44.4. Constant terms and the transfer

Let V, V' be spaces with strings for the same N over E, with $\operatorname{len} V' = \operatorname{len} V - 1$ or $\operatorname{len} V = \operatorname{len} V' = 0$, each with an adelic frame in which its Gram matrix is the same diagonal matrix of signs at every infinite place and the identity at the finite places; matrices are written in the bases of Definition 44.2.1. Let R, R' be the Bessel subgroups. Write $\mathbb{A} = \mathbb{A}_F$ for the groups and $\mathbb{A}_E$ for the matrices. Put $[R] := R(F)\backslash R(\mathbb{A})$ and $[U(V')] := U(V')(F)\backslash U(V')(\mathbb{A})$, with the quotient

measures of fixed Haar measures; $[R]$ is compact. Let $\mathbb{W} := \mathrm{Hom}(V, V')$, with the pairing $(y, y') \mapsto \mathrm{tr}(y'^{*}y)$. For each finite index set fix a continuous unitary character of the norm-one adeles of E trivial on the norm-one elements of E; for the index sets of the bases of V and V' these give the Weil representation ω of $U(V)(\mathbb{A})\times U(V')(\mathbb{A})$ on the Banach space $\mathcal{S}$ of data of [GL] (see [We], [Ku, §1]), with the Heisenberg operators $\mathrm{Heis}(y)$, $y \in \mathbb{W}(\mathbb{A}_E)$, and the theta functional $\theta\colon \mathcal{S} \to \mathbb{C}$. We use: $\omega(g,h)\mathrm{Heis}(y)\omega(g,h)^{-1} = \mathrm{Heis}(hyg^{-1})$; $\theta \circ \omega(\gamma, \gamma') = \theta$ for rational γ, γ'; $y \mapsto \theta(\mathrm{Heis}(y)\phi)$ is invariant under rational translations on every rational subspace of $\mathbb{W}$ on which the pairing vanishes. *Status*: classical.

Let d be weights on the basis of V' that change sign under the involution of the basis fixing the definite part and exchanging f'_i and $f'_{s'-1-i}$ ($s' = \mathrm{len}\, V'$), for instance $d = 0$ or the grading, and let $\mathbb{W}^d_{>0}$ be the maps whose entries of degree ≤ 0 vanish. The pairing vanishes on $\mathbb{W}^d_{>0}$. A map $V' \to V'$ has degree ≥ 0 for d if its entry in the row of e' and column of e'' vanishes whenever $d(e') < d(e'')$.

Definition 44.4.1. For a datum ϕ and a map $w \in \mathbb{W}(\mathbb{A}_E)$, the *constant term* of ϕ at w is the mean

$$\mathrm{CT}^d_w(\phi) := \int_{\mathbb{W}^d_{>0}(E)\backslash \mathbb{W}^d_{>0}(\mathbb{A}_E)} \theta\big(\mathrm{Heis}(y)\mathrm{Heis}(w)\phi\big)\, dy$$

for the probability Haar measure. We write $\mathrm{CT}_w := \mathrm{CT}^d_w$ when d is the grading of V'.

In the proof, for rational w, CT^d_w depends only on w modulo rational maps of degree ≥ 0. For the fixed spaces and weights d there is a constant $C_d \geq 0$ such that for every datum ϕ,

$$\theta(\phi) = \sum_w \mathrm{CT}^d_w(\phi), \qquad \sum_w |\mathrm{CT}^d_w(\phi)| \leq C_d \|\phi\|,$$

the sums over the classes of rational maps modulo maps of degree ≥ 0, with the norm of $\mathcal{S}$. This is the Fourier expansion along $\mathbb{W}^d_{>0}$ in a mixed model; compare [We]. *Status*: classical; the norm estimate is used in the form proved in the argument.

Definition 44.4.2. Let $\phi \in \mathcal{S}$, and let $\mathcal{W}$ be a continuous function on $R(\mathbb{A})$, left invariant under $R(F)$, with $\mathcal{W}(nr) = \psi_{\mathcal{U}}(n)\mathcal{W}(r)$ for all $n \in R(\mathbb{A})$ such that $n - 1$ has degree ≥ 2. The *transfer* of $\mathcal{W}$ is the function on $U(V')(\mathbb{A})$

$$\mathcal{W}'(h) := \int_{[R]} \mathcal{W}(r)\, \mathrm{CT}_{\mathrm{st}}\big(\omega(r,h)\phi\big)\, dr.$$

In the proof $\mathcal{W}'$ is continuous and left $R'(F)$-invariant. For each class w choose a rational representative $\tilde{w}$, and put $\mathcal{Q}^d_w(h,r) := \mathcal{W}(r)\, \mathrm{CT}^d_{\tilde{w}}(\omega(r,h)\phi)$; a class survives, or agrees with st, if $\tilde{w}$ does. In the proof, each $\mathcal{Q}^d_w$ is jointly continuous in (h,r), and for fixed h $\sum_w |\mathcal{Q}^d_w(h,r)|$ is bounded in r; and if $u \in R(\mathbb{A})$, $u - 1$ has degree ≥ 2 and $\tilde{w}(u-1)$ has degree ≥ 1, then

$$\mathcal{Q}^d_w(h, ur) = \frac{\psi_E\big(\frac{1}{2} j_E\, \mathrm{tr}(\mathrm{st}^{*}\mathrm{st}\, u)\big)}{\psi_E\big(\frac{1}{2} j_E\, \mathrm{tr}(\tilde{w}^{*}\tilde{w}\, u)\big)}\, \mathcal{Q}^d_w(h,r).$$

The ratio is 1 for all such u exactly when $\tilde{w}$ survives.

Problem 44.4.1. Let $\gamma \in R(F)$ and $\gamma' \in U(V')(F)$, and let γ' have degree ≥ 0 for the weights d (no condition when $d = 0$). Then $\mathcal{Q}^d_{\gamma' w \gamma^{-1}}(\gamma' h, \gamma r) = \mathcal{Q}^d_w(h, r)$.

We use special cases of two facts on discrete groups. Let G be a second-countable locally compact Hausdorff group with a left Haar measure, $\Gamma \subset G$ a discrete subgroup acting on a countable set $\mathcal{X}$, quotients by discrete subgroups carrying the induced measures, and $\mathfrak{k}_{\mathfrak{x}}$ ($\mathfrak{x} \in \mathcal{X}$) measurable functions on G with $\mathfrak{k}_{\gamma\mathfrak{x}}(\gamma g) = \mathfrak{k}_{\mathfrak{x}}(g)$. If $\sum_{\mathfrak{x}} |\mathfrak{k}_{\mathfrak{x}}|$ is integrable on $\Gamma\backslash G$, then $\mathfrak{k}_{\mathfrak{x}}$ is integrable on $\mathrm{Stab}_\Gamma(\mathfrak{x})\backslash G$ and $\int_{\Gamma\backslash G} \sum_{\mathfrak{x}' \in \Gamma\mathfrak{x}} \mathfrak{k}_{\mathfrak{x}'} = \int_{\mathrm{Stab}_\Gamma(\mathfrak{x})\backslash G} \mathfrak{k}_{\mathfrak{x}}$ [Bou]. And if $\mathfrak{f}$ is continuous, integrable on $\Gamma\backslash G$ and left Γ-invariant, with $\mathfrak{f}(ug) = \zeta \mathfrak{f}(g)$ for all g, one $u \in G$ and one $\zeta \neq 1$ of modulus one, then $\int_{\Gamma\backslash G} \mathfrak{f} = 0$; Problem 44.3.4 is the finite case. *Status*: the first is classical; the second is elementary, and is used in the form proved in the argument.

Problem 44.4.2. Let d be 0 or the grading of V', and let the class w fail the test for d. Then for every h, $\int_{[R]} \sum_{w' \in wR(F)} \mathcal{Q}^d_{w'}(h, r)\, dr = 0$.

Let w_+ and st_+ be the matrices of the columns of positive grade of w and of st. In the proof, if a rational w survives for $d = 0$, then w_+ has linearly independent columns and $\bar{w}_+^t J_{V'} w_+ = \bar{\mathrm{st}}_+^t J_{V'} \mathrm{st}_+$; Problem 44.3.1 illustrates this. Witt's theorem: if w_1, w_2 are matrices over E of the same shape, with linearly independent columns and $\bar{w}_1^t J w_1 = \bar{w}_2^t J w_2$ for a nondegenerate hermitian J, then $g w_1 = w_2$ for some invertible square g over E with $\bar{g}^t J g = J$ [Sch]. *Status*: classical.

Problem 44.4.3. Let w be a rational map with w_+ of linearly independent columns and $\bar{w}_+^t J_{V'} w_+ = \bar{\mathrm{st}}_+^t J_{V'} \mathrm{st}_+$. Then $\gamma' w$ agrees with st for some $\gamma' \in U(V')(F)$.

Let $\mathcal{C}$ be the set of classes for the weights 0 that agree with st. In the proof, if w agrees with st, $\gamma \in R(\mathbb{A})$ and $\gamma' \in U(V')(\mathbb{A})$, then $\gamma' w \gamma$ agrees with st if and only if $\gamma' \in R'(\mathbb{A})$; in particular $\mathcal{C}$ is stable under $w \mapsto w\gamma$, $\gamma \in R(F)$. Problem 44.3.2 illustrates this.

Problem 44.4.4. Let a group H act on a set $\mathcal{X}$, let $S \subseteq H$ be a subgroup and $\mathcal{C} \subseteq \mathcal{X}$ a subset such that for $\mathfrak{x} \in \mathcal{C}$, $g\mathfrak{x} \in \mathcal{C}$ if and only if $g \in S$. Then $(q, \mathfrak{x}) \mapsto q_{\mathrm{rep}}\mathfrak{x}$, for any choice of representatives q_{rep}, is a bijection from $(H/S) \times \mathcal{C}$ onto the union of the translates of $\mathcal{C}$.

The second selection takes place inside the period: in the proof, for every h,

$$\int_{[R]} \sum_{w \in \mathcal{C}} \mathcal{Q}^0_w(h, r)\, dr = \mathcal{W}'(h).$$

For each datum, the sum of the constant terms over $\mathcal{C}$ equals the sum over the classes agreeing with st for the grading of V'; every surviving representative of one of those lies in the class of st; st survives, and $R(F)$ fixes its class. Problem 44.3.3 illustrates this.

Problem 44.4.5. For every $h \in U(V')(\mathbb{A})$, the series on the right converges absolutely and

$$\int_{[R]} \mathcal{W}(r)\, \theta\big(\omega(r, h)\phi\big)\, dr = \sum_{q \in U(V')(F)/R'(F)} \mathcal{W}'(q_{\mathrm{rep}}^{-1} h).$$

Problem 44.4.6. Let $\Gamma' \subseteq \Gamma$ be discrete subgroups of G as above, $\mathfrak{f}$ a continuous left Γ'-invariant function and z a continuous left Γ-invariant function on G, with $\sum_{q\in\Gamma/\Gamma'}|\mathfrak{f}(q_{\mathrm{rep}}^{-1}h)z(h)|$ integrable on $\Gamma\backslash G$. Then $\mathfrak{f}z$ is integrable on $\Gamma'\backslash G$ and

$$\int_{\Gamma\backslash G}\sum_{q\in\Gamma/\Gamma'}\big(\mathfrak{f}(q_{\mathrm{rep}}^{-1}h)\,z(h)\big)\,dh=\int_{\Gamma'\backslash G}\mathfrak{f}(h)z(h)\,dh.$$

The height of $h\in U(V')(\mathbb{A})$ is the reciprocal of the first minimum, cut off at 1, of the lattice of the space moved by h. In the proof, for every g_0 and every large integer e, each h has a $\gamma'\in U(V')(F)$ with $\|\omega(g_0,\gamma'h)\|\le 2^e$ times the e-th power of the height of h; we use $g_0=1$. This growth estimate is proved by reduction theory; compare [KR]. A function z on $U(V')(\mathbb{A})$ is *automorphic* if it is continuous, left invariant under $U(V')(F)$ and invariant under right translation by a neighbourhood of 1 in the finite adelic group, and *rapidly decreasing* if for every integer $b\ge 0$, $|z(h)|$ times the b-th power of the height of h is bounded. For each rapidly decreasing z, in the proof, $\sum_{w\in\Sigma}|\mathcal{Q}_w^0(h,r)z(h)|\le C'$ for one constant C', every h, r and every set Σ of classes, and $\sum_q|\mathcal{W}'(q_{\mathrm{rep}}^{-1}h)z(h)|\le C'\,\mathrm{vol}[R]$, and $[U(V')]$ has finite volume. Problems 44.4.1–44.4.6, with the facts stated in this section, prove the following.

Theorem 44.4.1. *With E, the frames, ω and $\mathcal{S}$ as in this section, let V, V' be hermitian spaces over E, each the orthogonal sum of a definite part and a string for the same N, with* $\operatorname{len}V'=\operatorname{len}V-1$ *or* $\operatorname{len}V=\operatorname{len}V'=0$*; let R, R' be their Bessel subgroups and* st *the string map. Let $\phi\in\mathcal{S}$, $\mathcal{W}$ be as in Definition 44.4.2, and z be an automorphic, rapidly decreasing function on $U(V')(\mathbb{A})$. Then $\mathcal{W}'z$ is integrable on $R'(F)\backslash U(V')(\mathbb{A})$ and*

$$\int_{[R]}\mathcal{W}(r)\int_{[U(V')]}\theta\big(\omega(r,h)\phi\big)\,z(h)\,dh\,dr=\int_{R'(F)\backslash U(V')(\mathbb{A})}\mathcal{W}'(h)\,z(h)\,dh.$$

What has been proved. A Bessel period of a theta function, paired with a form on V', is the integral of the transfer against the form over $R'(F)\backslash U(V')(\mathbb{A})$. Used as given: the facts on ω, Heis and θ stated at the start of Section 44.4, the expansion of θ, the statements marked "in the proof", Witt's theorem and the facts on discrete groups. If $c_V=c_{V'}$, then in the proof, for each h, the function $r'\mapsto\mathcal{W}'(r'h)$ on $R'(\mathbb{A})$ satisfies the hypothesis on $\mathcal{W}$ for V': the transfer is again a Bessel function, on a string of length $\max(\operatorname{len}V-1,0)$. Along the lower chain $Y_0,\dots,Y_m$ the string of Y_j has length $m-1-j$ for $j<m$, and that of Y_m has length 0.

In the Lean formalization. Theorem 44.4.1 is `New.Grading.integral_besselGroup_unfold` in the Lean proof [GL]. The adelic definitions specialize `New.Grading`, `stringMap`, `constTerm`, `unfoldStep`, `Survives` and `HasDegreeGe`; the finite examples use the same matrix formulas. The expansion of θ is `hasSum_constTerm`, the matrix form of Witt's theorem is `New.witt`, and the growth estimate is `eventually_exists_norm_omega_le_mul_height_pow`. The statements marked "in the proof" are at these places of [GL], commit `72fead1`: the descriptions of $\mathcal{U}$ and R, compactness of $[R]$, the character $\psi_{\mathcal{U}}$ and st*st, the degree of st, class independence of CT_w^d, continuity and invariance of $\mathcal{W}'$, the terms $\mathcal{Q}_w^d$ and their phase law, the Gram condition, preservation of agreement, the second selection, the two estimates and the finite volume, the case $c_V=c_{V'}$, the chain gradings, the transposition, the orbit fact and the twisted integral.

Hints

§44.1.

44.1.2: Exchange the two sums and use $\sum_b \zeta_3^{b(Q(v)-1)}$.

44.1.3: -1 is not a square in $\mathbb{F}_3$, so Q is anisotropic.

44.1.4: Problems 44.1.2 and 44.1.3; substitute $h \mapsto \gamma h$, as in Problem 36.2.1.

§44.2.

44.2.1: Expand $\bar{u}^t J_V u = J_V$ entry by entry.

44.2.2: Compute the shift $\mathrm{st}^*\mathrm{st}$ on the string basis.

§44.3.

44.3.1: Which u are admissible for $v = 0$, and which for $v \neq 0$? Then Problem 44.2.2.

44.3.2: Count first, then second columns. The Bessel subgroup of a string of length two fixes f_0'.

44.3.3: Compare the entries used for agreement with those of negative degree; then compute the two phases.

§44.4.

44.4.1: Move $\omega(\gamma, \gamma')$ to the left through the Heisenberg operators, then substitute $y \mapsto \gamma' y \gamma^{-1}$ in the mean. An element of $R(\mathbb{A})$ preserves the span of the basis vectors of grade $\geq i$ for each i, so it does not lower grades (use unitarity for the grades < 0 and for the definite part).

44.4.2: Unfold the orbit with $\Gamma = R(F)$ and Problem 44.4.1; on the stabilizer quotient use the vanishing of twisted integrals.

44.4.3: Witt's theorem in V', for the columns of positive grade.

44.4.5: Expand θ for the weights 0, and sort the classes into the translates of $\mathcal{C}$ and the rest.

44.4.6: The orbit fact for Γ acting on Γ/Γ'; compare Problem 10.1.1.

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