TEST FORMS AND THE HEIGHT OF A HECKE RATIO

Prerequisites. Characters of finite abelian groups; $\mathbb{C}_\ell$, $|\ell|_\ell = 1/\ell$. Chapter 5, Section 5.3: isomorphisms $\iota \mid \ell$, $|x|_\iota := |\iota(x)|_\ell$, families and the height ht, Problems 5.3.2 and 5.3.3, and the Fock norm of Problem 5.4.4. Chapter 29: congruence modules. Chapter 67, for the existence of regular families only. Chapter 33: the data, s_{fin}, s_∞, h_{dir} and the budget $B = U/(mk)$ (Definition 33.4.1). Chapter 34: twists, $\mathcal{M}$, χ_M, υ, c_{vac}, and critical pairs $P = (\nu, z)$ with their ratios $R_P(\xi, \epsilon)$ (Definition 34.10.1). Chapter 48: local prime sums, and forms integral at η (Section 48.3). Chapter 61: false lifts and clearing. Chapter 62: rows λ_P (Definition 62.2.1), main pairs and empty rows (Definition 62.4.1), $\phi_{P,\xi}$, $\mathrm{alg}_P(\xi)$ (Definition 62.3.1) and Theorem 62.4.1. Chapter 63: $s^{\mathrm{tot}}_{\mathrm{fin},p}$, the twelve copies, Theorem 63.5.1.

This chapter closes the cost of one lift, whose statement chapter is Chapter 60. There the height of the family $Z_j(1)/Z_j(\theta_\iota)$, for a regular family θ of m-fold twists, needs among other inputs a height bound for $\iota \mapsto R_P(\vartheta_\iota)^{-1}$, $\vartheta_\iota := \theta_\iota^j$, for the pairs P main for j in the factorization of the Hecke value product. By Theorem 62.4.1, $R_P(\vartheta)^{-1} = (\mathrm{Nm}\, Q_\vartheta)^{2z-1}\|\phi_{P,1}\|^2/\|\phi_{P,\vartheta}\|^2$; the untwisted squared norm has a height bound, and $\mathrm{alg}_P(\vartheta)$ is integral at $\ell \nmid \mathrm{Nm}\, Q_\vartheta$. How does the proof bound $|\|\phi_{P,\vartheta_\iota}\|^2|_\iota$ from below? With *test forms*: the lower bound (1) for their pairings with ϕ_{P,ϑ_ι} combines the sharp jet of Chapter 63 with an auxiliary Hecke ratio. We begin with a line in $\mathbb{C}^2$.

64.1. Two operations on heights

A height bound $\mathrm{ht}(y) \le c$ for a family $y = (y_\iota)$ is as in Section 5.3; products and powers are treated in Problem 5.3.2. We need two more operations.

Problem 64.1.1. Let $H \neq 0$ and $\alpha \in \mathbb{R}$ with $|\sigma H| \le e^\alpha$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, and let $e(\ell) \in \mathbb{R}$, of any sign, one for each prime ℓ, with $|Hy_\iota|_\iota \le \exp(e(\ell))$ for all $\iota \mid \ell$ and $\alpha + \sum_{\ell \in F} e(\ell) \le c$ for every finite set F of primes. Then $\mathrm{ht}(y) \le c$.

Problem 64.1.2. If $\mathrm{ht}(y) \le c \le c''$ and $|y''_\iota|_\iota \le |y_\iota|_\iota$ for every $\iota \mid \ell$ and every prime ℓ, then $\mathrm{ht}(y'') \le c''$.

64.2. Two tests and a line

Let V be a finite-dimensional complex inner product space, conjugate-linear in the first variable, let $0 \neq v \in V$, and let pr be the orthogonal projection onto $\mathbb{C}v$.

Problem 64.2.1. For all $\Xi, \Xi' \in V$,

$$\langle \mathrm{pr}\,\Xi, \mathrm{pr}\,\Xi' \rangle = \frac{\langle \Xi, v\rangle\langle v, \Xi'\rangle}{\|v\|^2}.$$

Problem 64.2.2. In $V = \mathbb{C}^2$ with the standard inner product let $v = (1, 2)$, $\Xi = (1, 0)$ and $\Xi' = (0, 1)$. Then $\langle \Xi, v\rangle\langle v, \Xi'\rangle = 2$ and $\langle \mathrm{pr}\,\Xi, \mathrm{pr}\,\Xi'\rangle = 2/5$, so $|\langle \mathrm{pr}\,\Xi, \mathrm{pr}\,\Xi'\rangle|_\iota = 5$ for every $\iota \mid 5$, although v, Ξ and Ξ' have integral coordinates.

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So projecting integral vectors to a line can create a denominator, a congruence in the sense of Chapter 29. Given an upper bound for $|\langle \operatorname{pr}\Xi, \operatorname{pr}\Xi'\rangle|_\iota$, a lower bound for $|\langle \Xi, v\rangle\langle v, \Xi'\rangle|_\iota$ gives one for $|\|v\|^2|_\iota$. For a critical pair $P = (\nu, z)$ with $z \geq 2$, Theorem 62.4.1(i) states that for every tame character ξ

$$R_P(\xi) = \frac{\|\phi_{P,\xi}\|^2/(\operatorname{Nm} Q_\xi)^{2z-1}}{\|\phi_{P,1}\|^2}.$$

Problem 64.2.3. Let $P = (\nu, z)$ be a critical pair with $z \geq 2$, ϑ a twist, $\operatorname{pr}_{P,\vartheta}$ the orthogonal projection onto $\mathbb{C}\phi_{P,\vartheta}$, Ξ, Ξ' forms with $\langle \Xi, \phi_{P,\vartheta}\rangle\langle \phi_{P,\vartheta}, \Xi'\rangle \neq 0$, and P' a critical pair with $R_{P'}(\vartheta) \neq 0$. Then

$$R_P(\vartheta)^{-1} = (\operatorname{Nm} Q_\vartheta)^{2z-1} \cdot \|\phi_{P,1}\|^2 \cdot \langle \operatorname{pr}_{P,\vartheta}\Xi, \operatorname{pr}_{P,\vartheta}\Xi'\rangle \cdot R_{P'}(\vartheta)^{-2} \cdot \frac{R_{P'}(\vartheta)^2}{\langle \Xi, \phi_{P,\vartheta}\rangle\langle \phi_{P,\vartheta}, \Xi'\rangle}.$$

In the application P is main for j, $\vartheta = \theta_\iota^j$ for a regular family θ, Ξ and Ξ' are test forms and P' is ramified; Problem 64.4.5 bounds the five factors.

Definition 64.2.1. Let $P = (\nu, z)$ be a critical pair with $z \geq 2$, ϑ a twist and ℓ a prime. A *test form for* ℓ (for P and ϑ) is a form Ξ in the space of forms of $\phi_{P,\vartheta}$, on the definite space of rank $2z$ (Definition 62.2.2), such that:

(1) Ξ is integral at every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$ (Section 48.3);
(2) Ξ is orthogonal to the false lifts of the rank-one lift: for every finite place w of E with $\bar{w} \neq w$ that is not a place of the lift or the conjugate of one, Ξ is orthogonal to every form in the image of e_{X-1} that is killed by the Hecke polynomial at w at its test value (as in Definition 61.3.2) and by the descent, the adjoint of the lift. Here e_{X-1} is the idempotent with label $X - 1$ at every chosen place; since the source has rank one, it is the mean over $\prod_{w\in\mathcal{M}} k_w^\times$.

Let $\theta = (\theta_\iota)$ be a family of m-fold twists, one for every prime ℓ and every $\iota \mid \ell$.

Definition 64.2.2. The family θ has *small ratios* if (i) ℓ is prime to $\operatorname{Nm} Q_{\theta_\iota}$ for every $\iota \mid \ell$, and (ii) for every $1 \leq j \leq m$, every ramified critical pair P and $\epsilon = \pm 1$, the family $\iota \mapsto R_P(\theta_\iota^j, \epsilon)^{-1}$ has height at most x_1.

Definition 64.2.3. The family θ is *regular* if it has small ratios and (iii) for every $1 \leq j \leq m$ there are reals $\operatorname{cl}(\ell)$, one for each prime ℓ, with $\sum_{\ell\in F} \operatorname{cl}(\ell) \leq B$ for every finite set F of primes, such that for every $\iota \mid \ell$, with $\vartheta := \theta_\iota^j$: $\operatorname{cl}(\ell)$ clears the projection at (ι, ϑ, j) (Definition 61.3.4); and for every critical pair P main for j and any two test forms Ξ, Ξ' for ℓ for P and ϑ, $|\langle \operatorname{pr}_{P,\vartheta}\Xi, \operatorname{pr}_{P,\vartheta}\Xi'\rangle|_\iota \leq e^{\operatorname{cl}(\ell)}$.

That regular families exist is Theorem 67.2.1.

64.3. A large pairing from characters

Where does a test form with a large pairing come from? In the proof of (1) its pairing with $\phi_{P,\vartheta}$ is a scalar multiple of a normalized character sum of a theta kernel with twelve coordinates at each τ; after scaling, the value of the kernel at the identity is the coefficient estimated in Theorem 63.5.1. Two elementary facts carry the passage.

Problem 64.3.1. Let Cl be a finite abelian group of order N_c, $\operatorname{kn}\colon \mathrm{Cl} \to \mathbb{C}$, and for a character ψ of Cl let $\operatorname{av}_\psi := N_c^{-1} \sum_{g\in\mathrm{Cl}} \psi(g) \operatorname{kn}(g)$. Then $\sum_\psi \operatorname{av}_\psi = \operatorname{kn}(1)$,

the sum over all characters, and for every prime ℓ and every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$ some ψ has $|\eta(\mathrm{av}_\psi)| \geq |\eta(\mathrm{kn}(1))|$. If $N_c = 2$ and kn is the indicator of 1, both av_ψ equal $1/2$, of 2-adic absolute value $2 \geq |\mathrm{kn}(1)|_2$.

Problem 64.3.2. Let G be a finite group, $\mathcal{A}_3, \mathcal{A}_1\colon G \to \mathbb{C}$, and $L_* \in \mathbb{C}$ with $L_*^{\#G} = \prod_{\chi \in G} \mathcal{A}_3(\chi)\mathcal{A}_1(\chi)$. Let $\eta\colon \mathbb{C} \to F_0$ be a ring homomorphism into a field with an absolute value. Then for every $\psi \in G$ there are $\chi_3, \chi_1 \in G$ with $\chi_3\chi_1 = \psi$ and $|\eta(\mathcal{A}_3(\chi_3)\mathcal{A}_1(\chi_1))| \geq |\eta(L_*)|$. For $G = \{1, \psi\}$, $\mathcal{A}_3 = (1, 9)$, $\mathcal{A}_1 = (9, 1)$ and $L_* = 9$, exactly one factorization of ψ satisfies the inequality for $|\cdot|_3$.

For a main pair the proof of (1) splits the $2z$ coordinates into blocks of sizes 12, 4, 4 and $2z - 20$: the jet lives on the first block, the twist on the second, and the last two carry auxiliary lifts.

Problem 64.3.3. For an integer $z \geq 2$, $12 + 4 + 4 + \max(2z - 20, 0) = 2z$ if and only if $z \geq 10$. For a pair main for j, $10000 \leq m < z \leq r$ and $4 \leq 2z - 20 \leq t$.

Let P be main for j, $\lambda := \lambda_P$ its row, and $\mathbf{h} = (h_{\tau,i})$ a multi-index on the twelve copies with $\sum_i h_{\tau,i} = \lambda(\tau)$ for every τ (Chapter 63). For the twelve coordinates of the first block the proof uses an integer vector in the Weyl representation of the lift; over every field of characteristic zero it is fixed by every family of invertible matrices fixing those twelve basis vectors at each τ, and its Fock polynomial in these coordinates is $T_{\mathbf{h}} X^{\mathbf{h}}$, with

$$T_{\mathbf{h}} := \prod_{\tau \in \Phi} \frac{\lambda(\tau)!}{\prod_{i=1}^{12} h_{\tau,i}!}.$$

This is read from the code [GL]. Give the polynomials in the $X_{\tau,i}$ the Fock inner product of Problem 5.4.4 in each variable: $\|X^{\mathbf{h}}\|^2 = \prod_{\tau,i} h_{\tau,i}!$, and monomials are orthogonal.

Problem 64.3.4. $T_{\mathbf{h}}$ is the coefficient of $X^{\mathbf{h}}$ in $\prod_\tau (X_{\tau,1} + \cdots + X_{\tau,12})^{\lambda(\tau)}$, and $T_{\mathbf{h}} \in \mathbb{Z}$. For a polynomial $f = \sum_{\mathbf{h}'} a_{\mathbf{h}'} X^{\mathbf{h}'}$, $\langle T_{\mathbf{h}} X^{\mathbf{h}}, f \rangle = \prod_\tau \lambda(\tau)! \cdot a_{\mathbf{h}}$. For one place, $\lambda = 13$ and $\mathbf{h} = (2, 1, \ldots, 1)$, $T_{\mathbf{h}} = 13!/2$.

Theorem 63.5.1, for λ with $\kappa(\tau) \leq \lambda(\tau) \leq \kappa(\tau) + x^{42}$ for every τ, gives $c_{\mathrm{att}} \in \mathbb{C}^\times$ with $|\sigma(c_{\mathrm{att}})| \leq \exp(fmb/k)$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that for every modulus $\mathfrak{m}$ of E with ideal (π_M), every prime p and every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_p$ there are $\mathbf{h}$ with $\sum_i h_{\tau,i} = \lambda(\tau)$ for every τ and $y_0 \in (\mathcal{O}_E/\mathfrak{m})^{12}$ with

$$\mathrm{val}\Big(\eta\Big(\frac{c_{\mathrm{vac}}^{12}}{c_{\mathrm{att}}} \prod_{\tau \in \Phi} \lambda(\tau)! \cdot [z^{\mathbf{h}}]\mathrm{Jet}_{y_0}\Big)\Big) \leq s_{\mathrm{fin},p}^{\mathrm{tot}}.$$

For a pair P main for j, the row λ_P satisfies these bounds (Problem 62.4.2).

Now the objects read from the proof [GL]. In this sheet $G_{\mathcal{M}}$ is the finite group of pairs ψ consisting of a continuous unitary character of the norm-one ideles of E, trivial on the norm-one elements of $E^\times$, of weight 0, and its tame part, which is trivial off $\mathcal{M}$; $\psi(g)$ is the value of the character at the norm-one idele given by a finite adelic point g of $U(1)$. For every $\psi \in G_{\mathcal{M}}$ and every even n with $4 \leq n \leq t$ there is a critical pair $(\nu, n/2)$ with empty row, of *rank* n (the rank $2z$ of the space of its lift, Definition 62.2.2), with the tame part of ψ, whose character becomes that of ψ after multiplication by $v^{n/2}$. For a pair $P = (\nu_P, z)$ main for j, let $P_\psi^{(4)}$ and

$P_\psi^{(2z-20)}$ be such pairs of ranks 4 and $2z-20$, and fix L_* with

$$L_*^{\#G_{\mathcal{M}}} = \prod_{\psi \in G_{\mathcal{M}}} \operatorname{alg}_{P_\psi^{(4)}}(1) \operatorname{alg}_{P_\psi^{(2z-20)}}(1).$$

For a pair P_0 with empty row, $\operatorname{alg}_{P_0}(1)$ is a positive real number (Chapter 62), and

$$|\sigma(\operatorname{alg}_{P_0}(1))| \le e^{x^{34}} \qquad \text{for every } \sigma \in \operatorname{Aut}(\mathbb{C});$$

this bound is read from the code [GL] and not proved in this workbook. Hence $L_* \neq 0$ and $|\sigma(L_*)| \le e^{2x^{34}}$ for every $\sigma \in \operatorname{Aut}(\mathbb{C})$. The classes of the finite adelic points of $U(1)$ modulo the rational points and the principal level at the chosen places form a finite abelian group Cl, of order N_c; every character of Cl has the form $g \mapsto \psi(g)$ for some $\psi \in G_{\mathcal{M}}$. Fix representatives g_c of the classes $c \in \mathrm{Cl}$.

The exact pairing. Let P be main for j, P' a critical pair with empty row of rank four, ℓ a prime, $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism and ϑ a twist with $\ell \nmid \operatorname{Nm} Q_\vartheta$. Let $\mathbf{h}$ and y_0 be given by Theorem 63.5.1 for $\lambda = \lambda_P$, (ℓ, η) and the modulus formed by the chosen places, with exponent one, whose ideal is (π_M). Write $\nu_{P'}$ for the character of P' and put $\chi_0 := (\nu_P/\nu_{P'})\, \upsilon^{z-2}$. Let $\mathrm{kn}(g)$ be the scalar theta kernel on the twelve-block, with characters 1 and χ_0, polynomial $X^{\mathbf{h}}$ and the indicator of y_0, evaluated at the point of $U(1)$ with finite part g and trivial archimedean part, and at the identity of $U(12)$. There is a number $c_{\mathrm{kn}} \neq 0$ such that $c_{\mathrm{kn}} \operatorname{kn}(1) = c_{\mathrm{vac}}^{12} c_{\mathrm{att}}^{-1} \prod_\tau \lambda_P(\tau)! \cdot [z^{\mathbf{h}}]\mathrm{Jet}_{y_0}$, so that $|\eta(c_{\mathrm{kn}} \operatorname{kn}(1))| \ge \ell^{-s^{\mathrm{tot}}_{\mathrm{fin},\ell}}$; and for every $\psi \in G_{\mathcal{M}}$, every factorization $\psi = \psi_3\psi_1$ in $G_{\mathcal{M}}$ and every finite adelic point g_0 of $U(1)$ there is a test form Ξ for ℓ for P and ϑ with

$$N_c^2 \langle \Xi, \phi_{P,\vartheta} \rangle = c_{\mathrm{att}} \operatorname{alg}_{P'}(1) \operatorname{alg}_{\psi_3,\psi_1} R_{P'}(\vartheta)\, c_{\mathrm{kn}} \operatorname{av}_{\psi,g_0}, \qquad \operatorname{av}_{\psi,g_0} := \frac{1}{N_c} \sum_{c \in \mathrm{Cl}} \psi(g_c g_0) \operatorname{kn}(g_c g_0),$$

where $\operatorname{alg}_{\psi_3,\psi_1} := \operatorname{alg}_{P_{\psi_3}^{(4)}}(1) \operatorname{alg}_{P_{\psi_1}^{(2z-20)}}(1)$. Here Ξ is a class sum, against the lift of P' at ϑ and the untwisted lifts of the two auxiliary pairs, of a point form with the integer vector above. *Status*: read from the code [GL] (see the remark at the end); not proved in this workbook.

Problem 64.3.5. Let P be main for j and P' a critical pair with empty row of rank four, and put $c_{\mathrm{test}} := c_{\mathrm{att}} \operatorname{alg}_{P'}(1)\, L_*$. Then $c_{\mathrm{test}} \neq 0$, $|\sigma(c_{\mathrm{test}})| \le \exp(fmb/k + 3x^{34})$ for every $\sigma \in \operatorname{Aut}(\mathbb{C})$, and for every prime ℓ, every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$ and every twist ϑ with $\ell \nmid \operatorname{Nm} Q_\vartheta$ there is a test form Ξ for ℓ for P and ϑ with

$$|\eta(c_{\mathrm{test}}\, R_{P'}(\vartheta))| \cdot \ell^{-s^{\mathrm{tot}}_{\mathrm{fin},\ell}} \le |\eta\langle \Xi, \phi_{P,\vartheta} \rangle|. \tag{1}$$

64.4. The height of a Hecke ratio

Let θ be regular, $1 \le j \le m$, $P = (\nu, z)$ a critical pair main for j, and put $\vartheta_\iota := \theta_\iota^j$. Fix a ramified critical pair P' with empty row of rank four; the proof takes one whose tame part is that of χ_M and whose character becomes χ_M after multiplication by υ^2. Let c_{test} be as in Problem 64.3.5 for P and P'. The ratio $R_{P'}(\vartheta_\iota)$ is a positive real number.

Problem 64.4.1. Let $\iota \mid \ell$. There are test forms Ξ_ι, Ξ'_ι for ℓ for P and ϑ_ι such that, with $y_\iota := \langle \Xi_\iota, \phi_{P,\vartheta_\iota} \rangle \langle \phi_{P,\vartheta_\iota}, \Xi'_\iota \rangle$,

$$|c_{\mathrm{test}}\, c_{\mathrm{test}}^{-}\, R_{P'}(\vartheta_\iota)^2|_\iota \cdot \ell^{-2s^{\mathrm{tot}}_{\mathrm{fin},\ell}} \le |y_\iota|_\iota;$$

in particular $y_\iota \neq 0$.

For each ι fix such Ξ_ι, Ξ'_ι, and let y_ι be the resulting product.

Problem 64.4.2. For every finite set F of primes,

$$\sum_{p\in F} 2\, s^{\mathrm{tot}}_{\mathrm{fin},p} \log p \le \sum_{\tau\in\Phi} s_{\mathrm{fin}}(\kappa(\tau),\tau).$$

Problem 64.4.3. For the data, $fmb/k + 3x^{34} \le B$ and $x_1 \le B$.

Problem 64.4.4. The family $\iota \mapsto R_{P'}(\vartheta_\iota)^2/y_\iota$ has height at most $2B + \sum_{\tau\in\Phi} s_{\mathrm{fin}}(\kappa(\tau),\tau)$.

Problem 64.4.5. In Problem 64.2.3 at ϑ_ι, with Ξ_ι, Ξ'_ι and P', the five factors have $|(\mathrm{Nm}\, Q_{\vartheta_\iota})^{2z-1}|_\iota = 1$ and, as families in ι, heights at most $B - \sum_\tau s_\infty(\kappa(\tau))$, B, $2x_1$ and $2B + \sum_\tau s_{\mathrm{fin}}(\kappa(\tau),\tau)$ for the last four. The sum of these four bounds is $4B + 2x_1 - h_{\mathrm{dir}} \le 6B - h_{\mathrm{dir}}$.

Problems 64.2.3 and 64.4.1–64.4.5 prove the following.

Theorem 64.4.1. *Let θ be regular, $1 \le j \le m$, and P a critical pair main for j. Then the family $\iota \mapsto R_P(\theta^j_\iota)^{-1}$ has height at most $6B - h_{\mathrm{dir}}$.*

Problem 64.4.6. For the data, $6B < h_{\mathrm{dir}}$. So the bound of Theorem 64.4.1 is negative.

We have proved Theorem 64.4.1 from Theorems 62.4.1 and 63.5.1 and the exact pairing of Section 64.3, taken as an input with the auxiliary pairs, the conjugate bound for their algebraic parts, the integer vector on the twelve-block, and the finite class group Cl with the lifting of its characters to $G_{\mathcal{M}}$; the bounds on ramified ratios and on denominators of projections enter as the definition of regular families. By Problem 62.5.1, at ϑ_ι the inverse ratio of Hecke value products is, up to a unit at ℓ, the square of a product of $k/2$ factors $R_P(\vartheta_\iota)^{-1}$, P main, and $5m$ factors $R_{P'}(\vartheta_\iota,\epsilon)^{-1}$, P' ramified, with signs depending only on j; so its height is at most $k(6B - h_{\mathrm{dir}}) + 10mx_1$ (Problem 60.2.2).

In the Lean formalization. In the Lean proof [GL] (`grh/src/New/`), Theorem 64.4.1 is `Data.heightLe_inv_ratio`, whose proof also chooses P'; (1) is `Data.exists_isTestForm_le_norm_inner_trivialLift`, in whose proof are the group $G_{\mathcal{M}}$, the blocks, the classes Cl and the value $c_{\mathrm{kn}}\,\mathrm{kn}(1)$ (step `h_lower`); its step `h_inner` is the exact pairing for the choices made there. The exact pairing in general follows from `Data.exists_isTestForm_inner_kernelForm_eq`, `Data.modKernel_div_apply` and `LValueData.trivialLift_eq_stdClassSum_translate`. Test forms are `DescentPair.IsTestForm`; Definitions 64.2.2 and 64.2.3 are `Data.HasSmallRatios` and `Data.IsRegularFamily`. The auxiliary pairs come from `Data.exists_hasEmptyRow`, the bound $e^{x^{34}}$ is `Data.CriticalPair.norm_map_integralAlgebraicPart_le`, L_* is from `Data.exists_root_prod_integralAlgebraicPart`, the integer vector is `Rows.exists_int_weylRep_invariant_fockPoly_eq_smul_monomial`, and the lifting of characters of Cl is in `exists_norm_le_norm_twisted_classSum`.

Hints

§64.1.

64.1.1: Only the primes with $e(\ell) > 0$ cost anything.

64.1.2: Keep the multiplier H.

§64.2.

64.2.1: $\operatorname{pr}\Xi = (\langle v, \Xi\rangle/\|v\|^2)\, v$, and the first slot is conjugate-linear.

64.2.3: Theorem 62.4.1(i) at $\xi = \vartheta$, and Problem 64.2.1 for $\|\phi_{P,\vartheta}\|^2$.

§64.3.

64.3.1: $\sum_\psi \psi(g)$ is N_c or 0. For the second statement, $\mathbb{C}_\ell$ is ultrametric.

64.3.2: For fixed ψ, compare the product of $|\eta(\mathcal{A}_3(\psi\chi^{-1})\mathcal{A}_1(\chi))|$ over χ with its largest factor.

64.3.4: The multinomial theorem, and Problem 5.4.4 variable by variable.

64.3.5: Translate the representative of the identity class to 1. Which character and which factorization give the two lower bounds? Use $N_c \in \mathbb{Z}$ to control its local absolute value.

§64.4.

64.4.1: Keep the twist fixed. Which two maps $\mathbb{C} \to \mathbb{C}_\ell$ account for the two orders of the arguments of the inner product?

64.4.2: Problems 48.1.4 and 33.4.2; all terms are nonnegative.

64.4.3: $B = fxb\log^3 b/(mk)$. Use Definition 33.3.1 and Problem 33.3.2: $m^2 \le 4x$, $k \le r \le 3x^5$, $x^{10^6} \le x_1$ and $x_1^{10^6} \le b$.

64.4.4: Problem 64.1.1: which multiplier works at both embeddings? Then Problems 64.4.1, 64.4.2 and 64.4.3.

64.4.5: Match the four factors with Theorem 62.4.1(iii), clause (iii) of Definition 64.2.3 with Problem 5.3.3(a), clause (ii) of Definition 64.2.2 for P' squared (Problem 5.3.2) and Problem 64.4.4; then Problem 64.1.2.

64.4.6: $h_{\mathrm{dir}} \ge b\,\#\Phi_a$ (Problem 33.4.5), $f = \#\Phi_a\,\varphi(q_0)$ (Problem 33.2.2), and compare $6\varphi(q_0)x\log^3 b$ with mk.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.