THE BOUND IS ATTAINED AT MANY PLACES

Prerequisites. Chapter 19: the least valuation $\mu(n)$ on the lemniscate, Theorems 19.2.1 and 19.3.1 with the series $y(z)$ and the ring $\mathfrak{o}[[y]]$ of Section 19.3, and Problems 19.1.1, 19.3.1, 19.3.2, 19.4.1, 19.4.2 and 19.4.3. The moments $t(a,w)$ of Section 15.1 and Theorem 16.3.1. The data (Definition 33.3.1, the constant v_0 of Section 33.3), the integral vacuum (Section 34.8), the local prime sums $s_{\mathrm{fin},p}$, the numbers c_p and the sets I_p (Section 48.1), and the jet series of a datum with its scale c_{jet} (Definition 50.4.3). Lagrange's four-square theorem; the ultrametric inequality in $\mathbb{C}_p$, whose valuation val has $\mathrm{val}(p)=1$.

Chapter 60 bounds the cost of one lift using the test forms of Chapter 64. At a prime p prime to the norm of the twist, a test form uses a Taylor coefficient on twelve copies of the variables; multiplied by $c_{\mathrm{vac}}^{12}\prod_\tau \lambda(\tau)!$ and divided by one c_{att} independent of p, it must have valuation at most the divisibility $s_{\mathrm{fin},p}^{\mathrm{tot}}$ predicted by the prime sum. How can one normalization serve all primes, when the partners $[-p^e]\tau$ of a place lower the prediction?

63.1. One curve, one partner

Recall $\mu(n)\le\gamma_p n$, $\gamma_p=p/(p^2-1)$, for $p\equiv 3 \pmod 4$ (Theorem 19.2.1).

Problem 63.1.1. For $p=3$: $\mu(3),\mu(9),\mu(11),\mu(81)=1,3,4,30$, each below $3n/8$.

For $p\equiv 3 \pmod 4$, $w\ge 0$ and $G\in\mathbb{Z}_p[\mathrm{i}][[X_1,X_2]]$ with $X_1=\mathrm{sl}\,s$, $X_2=\mathrm{sl}\,t$, the valuation of $\partial_s^{pw}\partial_t^{w}G(0,0)$ is at least $\mu(pw)+\mu(w)$ (Problem 19.4.2). A non-integral series does better.

Problem 63.1.2. $\partial_s^{pw}\partial_t^{w}e^{s^pt/p}\big|_{(0,0)}=(pw)!/p^w$, of valuation $v_p(w!)$. For $p=3$, $w=2$ this is 80, a unit, while for every $G\in\mathbb{Z}_3[\mathrm{i}][[X_1,X_2]]$, with $X_1=\mathrm{sl}\,s$, $X_2=\mathrm{sl}\,t$, the valuation of $\partial_s^6\partial_t^2G(0,0)$ is at least 2.

For a prime $p\equiv 3 \pmod 8$ and integers $a,w\ge 0$, Theorem 16.3.1 gives in particular $v_p(t(a,w))\ge\sum_{i\ge1,\ i\ \mathrm{odd}}(\lfloor a/p^i\rfloor-w)_+$ for the moments of Section 15.1, where w is the order of the derivative in the partner direction.

Problem 63.1.3. $v_{11}(t(27,1))=v_{11}(t(38,2))=v_{11}(t(49,3))=1$, and in each case the bound of Theorem 16.3.1 at $p=11$ is 1: in these three cases the theta function itself attains the bound.

Problem 63.1.4. At $p=3$: $v_3(t(a,1))=6,6,9$ for $a=11,19,27$, while the bound of Theorem 16.3.1 is $2,5,8$.

So a fixed theta function can miss. For each p and η the proof chooses twelve sections, one on each copy of the variables $(z_\tau)_{\tau\in\Phi}$.

Date: November 26, 2026.

63.2. What the partners take away

From now on we use the data (Definition 33.3.1). Fix a prime p and a ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_p$. Then c_p, I_p and the local prime sum $s_{\mathrm{fin},p}(y,\sigma) = c_p \sum_{e\in I_p} (y/p^e - \kappa([-p^e]\sigma))_+$ are those of Section 48.1. Let $\gamma_p := c_p \sum_{e\in I_p} p^{-e}$ (values in Problem 19.3.1). The *total local prime sum* is

$$s^{\mathrm{tot}}_{\mathrm{fin},p} := \sum_{\tau\in\Phi} s_{\mathrm{fin},p}(\kappa(\tau),\tau). \tag{1}$$

By Problem 19.4.1, $s^{\mathrm{tot}}_{\mathrm{fin},p} \le \gamma_p \sum_\tau \kappa(\tau)$. The difference is what the partners take away.

Problem 63.2.1. $\gamma_p \sum_\tau \kappa(\tau) - s^{\mathrm{tot}}_{\mathrm{fin},p} = c_p \sum_{\tau\in\Phi} \sum_{e\in I_p} \min\bigl(\kappa(\tau)/p^e,\ \kappa([-p^e]\tau)\bigr)$.

At an inert prime the partners enter the theta function through couplings. The proof uses a fixed curve: the lattice $\Lambda = \Omega_0 \mathcal{O}_K$, with Ω_0 as in Definition 50.4.3, with t, a_1 and the series y as in Section 19.3; let $\mathrm{area}(\Lambda)$ be the area of $\mathbb{C}/\Lambda$. Let σ_Λ be the odd entire function with $\sigma'_\Lambda(0) = 1$ and $\sigma_\Lambda(z+l) = \epsilon_l \exp\bigl(\pi(\bar l z + |l|^2/2)/\mathrm{area}(\Lambda)\bigr)\sigma_\Lambda(z)$ for $l \in \Lambda$, where $\epsilon_l = 1$ on 2Λ and -1 elsewhere. In this sheet let

$$\mathrm{th}(z) := e^{-\pi z \bar t/\,\mathrm{area}(\Lambda)}\, \frac{\sigma_\Lambda(z+t)}{\sigma_\Lambda(t)},$$

We read its Taylor series in $\mathbb{C}_p$ through η. For linear forms μ_β on $\mathbb{C}_p^\Phi$ and integers $c_\beta \ge 0$ (β in a finite set) put $\mathrm{th}_{\mathrm{fr}}(z) := \prod_\beta \mathrm{th}(\mu_\beta z)^{c_\beta}$, and for odd e let $\Gamma_e := \sum_\tau z_\tau^{p^e} z_{[-p^e]\tau}$ be the coupling of each place with its partner.

Normal form. Let p be inert in K, and let the forms be those of a tight frame (Definition 63.3.1 below). The proof shows that there are $\varpi \in \mathbb{C}_p$ with $\mathrm{val}\,\varpi = 1/(p^2-1)$, integral $\mathfrak{l}_{e,j}$ (e odd, $j \ge 0$) with $\mathfrak{l}_{e,0}$ a unit, and an integral series W in the z_τ, such that

$$\mathrm{th}_{\mathrm{fr}}(\varpi z) = W(z) \exp\Bigl(\frac{\varpi^{p+1}}{p} \sum_{e\ \mathrm{odd}} \sum_{j\ge 0} \mathfrak{l}_{e,j}\, \Gamma_e^{p^j}\Bigr).$$

Status: read from the proof.

Problem 63.2.2. A two-variable model: $K = \mathbb{Q}(\sqrt{-7})$, $p = 13$, variables z_1, z_2, $\mathrm{val}\,\varpi = 1/168$ and $\mathfrak{l}_{1,0}$ a unit. Then 13 is inert in K, $\mathrm{val}(\varpi^{14}/13) = -11/12$, and the coefficient of $z_1^{13} z_2$ in $(\varpi^{14}/13)\mathfrak{l}_{1,0}(z_1/\varpi)^{13}(z_2/\varpi)$ is $\mathfrak{l}_{1,0}/13$, of valuation -1; times $13!\,1!$ it has valuation 0, whereas the rate of one variable in degree 14 is $\gamma_{13}\cdot 14 = 13/12$.

63.3. Six copies along an isotropic vector

Definition 63.3.1. A *tight frame* of length M consists of elements $w_\beta \in E$ and integers $c_\beta \ge 0$, β in a finite set, with $\sum_\beta c_\beta\, \tau'(w_\beta)\overline{\tau(w_\beta)} = M\delta_{\tau'\tau}$ for $\tau,\tau' \in \Phi$, each w_β lying in the lattice $\mathfrak{a}_0 := E \cap \alpha_0 \hat{\mathcal{O}}_E$, α_0 the finite idele of the *Weil data* fixed with E in the proof. At η its forms are $\mu_\beta(z) := \sum_{\tau\in\Phi} \eta\bigl(c_{\mathrm{per}}\overline{\tau(w_\beta)}\bigr) z_\tau$, where c_{per} is the period factor of Definition 50.4.3 (not the Ω of Chapter 33). The *twelve copies* are the variables $z_{\tau,i}$, $\tau \in \Phi$, $1 \le i \le 12$, one coordinate of $\mathrm{Hom}(E, E^{12})$ at each infinite place and each of the twelve basis vectors; a multi-index on them is written $\mathbf{h} = (h_{\tau,i})$, and n copies are the variables $z_{\tau,i}$ with $i \le n$.

Tight frames. There is a tight frame of length M with $c_\beta \ge 6fb$ for every β such that for every prime p and every $\eta\colon \mathbb{C} \to \mathbb{C}_p$ some form μ_{β_0} has all its coefficients units. *Status*: read from the proof. Fix one; write $\mathfrak{o}$ for the valuation ring of $\mathbb{C}_p$, and $\mathfrak{o}[[y_{\mathrm{fr}}]]$ for the ring of series $G\big((y(\mu_\beta z))_\beta\big)$, G a power series over $\mathfrak{o}$ in one variable for each β.

Definition 63.3.2. Let $n \in \mathbb{N}$, ν a multi-index on Φ and $\rho \in \mathbb{R}$. We write $\mathrm{lv}_n(\nu) \le \rho$ if there are $H_1, \dots, H_n \in \mathfrak{o}[[y_{\mathrm{fr}}]]$ and multi-indices $\nu_1 + \dots + \nu_n = \nu$ with $\sum_{i=1}^n \mathrm{val}\big([z^{\nu_i}](H_i \,\mathrm{th}_{\mathrm{fr}})\big) \le \rho$.

The next problems use six scalings to make the product of the theta factors a unit of $\mathfrak{o}[[y_{\mathrm{fr}}]]$.
Isotropic scalings. Write $\mathrm{th} = \exp(-g)$ with $g(0) = 0$. For every $\delta \in \mathbb{Z}_p$ the series $\exp\big(\delta^2 g(z) - g(\delta z)\big)$ lies in $\mathfrak{o}[[y]]$. *Status*: read from the proof. For $0 \ne \alpha \in \mathcal{O}_K$, multiplication by α is an isogeny of degree $\alpha\bar\alpha$ of $\mathbb{C}/\Lambda$ (complex multiplication; Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Ch. II). For every $\alpha \in \mathcal{O}_K$ the proof shows $\mathrm{th}(\alpha z) = \mathrm{th}(z)^{\alpha\bar\alpha} v$ with $v \in \mathfrak{o}[[y]]$, that is, $\exp(\alpha\bar\alpha g(z) - g(\alpha z)) \in \mathfrak{o}[[y]]$; it then passes to p-adic limits of the pairs $(\alpha, \alpha\bar\alpha)$, among which are the pairs (δ, δ^2), $\delta \in \mathbb{Z}_p$.

Problem 63.3.1. $1 + 70^2 = 29 \cdot 13^2$; $(1, 70, 0, 0, 0, 0)$ is isotropic modulo 13^2; there is $\delta_2 \in \mathbb{Z}_{13}$ with $\delta_2 \equiv 70 \pmod{169}$ and $\delta_2^2 = -1$. For every prime p there are $\delta_2, \dots, \delta_5 \in \mathbb{Z}_p$ such that $\delta := (1, \delta_2, \delta_3, \delta_4, \delta_5, 0)$ has $\sum_{i=1}^6 \delta_i^2 = 0$.

Problem 63.3.2. If $\delta \in \mathbb{Z}_p^6$ and $\sum_i \delta_i^2 = 0$, then $\prod_i \mathrm{th}(\delta_i z)$ lies in $\mathfrak{o}[[y]]$ with constant term 1, and $\prod_i \mathrm{th}_{\mathrm{fr}}(\delta_i z)$ is a unit of $\mathfrak{o}[[y_{\mathrm{fr}}]]$.

Problem 63.3.3. If μ_{β_0} has unit coefficients, then $\mathrm{lv}_6(\nu) \le \gamma_p|\nu| - \sum_\tau \mathrm{val}(\nu_\tau!)$ for every ν. For $\nu = (3, 6)$ on two places and $p = 3$: $\binom{9}{3} = 84$, $v_3(9!) = 4$ and $v_3(3!\,6!) = 3$.

Two more six-copy bounds are read from the proof. At an inert prime p, integral combinations of the forms give each coordinate times a unit, and then $\mathrm{lv}_6(\nu) \le 0$ for every ν. At an inert prime $p > 2$, from the normal form: let $\Phi_0 \subseteq \Phi$, odd positive integers e_τ for $\tau \in \Phi_0$ such that no (τ_1, e') with $\tau_1 \in \Phi_0$, $e' \le e_\tau$ an odd positive integer and $(\tau_1, e') \ne (\tau, e_\tau)$ has $[-p^{e'}]\tau_1 = [-p^{e_\tau}]\tau$, integers $j_\tau \ge 0$, $\nu' := \sum_{\tau\in\Phi_0} p^{e_\tau} j_\tau[\tau]$, $\nu'' := \sum_{\tau\in\Phi_0} j_\tau[[-p^{e_\tau}]\tau]$ and $J := \sum_{\tau\in\Phi_0} j_\tau$; then

$$\mathrm{lv}_6(\nu' + \nu'') \le \frac{J}{p-1} - J - \frac{|\nu' + \nu''|}{p^2 - 1} - \mathrm{val}(\nu''!),$$

where $[\tau]$ is the multi-index with a single 1 at τ and $\nu''! := \prod_\tau \nu''_\tau!$.

63.4. Near pairs

We now describe the pairs used at non-split primes $p \le k$. Let $l_0(s)$ be as in Definition 33.3.1.

Definition 63.4.1. Let p be a prime. A pair (τ, e) with $\tau \in \Phi$ is *near* at p if $e \in I_p$, $2e < l_0(s)$, $\tau \in \Phi_b$ and $[-p^e]\tau \in \Phi_b$; its *start* is τ and its *end* is $[-p^e]\tau$.

A model: the group $(\mathbb{Z}/13)^\times$ in place of Φ, $p = 3$ treated as inert, the cutoff $2e < 3$, and the heavy set $\{1, 10\}$ in place of Φ_b.

Problem 63.4.1. The iterates of $[-3]$ of orders 0 to 4 from 1 and from 10 are $(1, 10, 9, 12, 3)$ and $(10, 9, 12, 3, 4)$; the only near pair is $(1, 1)$, with end 10.

Problem 63.4.2. With weight 27 on $\{1, 10\}$ and 1 elsewhere, $\sum_\tau \sum_{e \text{ odd}} (\kappa(\tau)/3^e - \kappa([-3^e]\tau))_+ = 8$, the only nonzero term being $\tau = 10$, $e = 1$; the near pair $(1, 1)$ contributes 0.

Problem 63.4.3. Let $p \le k$ be a prime with $\chi_K(p) \ne 1$. (a) Two distinct near pairs at p have distinct starts and distinct ends, and the end of a near pair is never the start of one. (b) If $p = d$, there are no near pairs. Hence the near pairs at a prime $p \le k$ satisfy the hypothesis on Φ_0 in Section 63.3.

63.5. Twelve copies and the theorem

On twelve copies the proof combines the two kinds of six-copy bounds: at a split prime, Problem 63.3.3 on six copies and the constant coefficient in degree zero on the other six; at an inert $p > k$, every place coupled with its partner for $e = 1$ on six copies and the span bound on six; at a non-split $p \le k$, the near pairs on six copies and Problem 63.3.3 on six. For $\lambda\colon \Phi \to \mathbb{N}$ let $u := \lceil 6b/k \rceil$ and

$$\Pi_\tau := \prod_{i=1}^{u} \big((\lambda(\tau) - u)_+ + i \big),$$

a product of u consecutive integers; the reserved u pays for the rounding. The proof shows: if $\kappa(\tau) \le \lambda(\tau) \le \kappa(\tau) + x^{42}$ for every τ, then at every prime p and every η

$$\mathrm{lv}_{12}(\lambda) \le s^{\mathrm{tot}}_{\mathrm{fin},p} + 2 \sum_\tau \mathrm{val}(\Pi_\tau) - \sum_\tau \mathrm{val}(\lambda(\tau)!).$$

Status: read from the proof. The loss added to $s^{\mathrm{tot}}_{\mathrm{fin},p} - \sum_\tau \mathrm{val}(\lambda(\tau)!)$ is zero at split primes, $\sum_\tau \mathrm{val}(\Pi_\tau)$ at non-split $p \le k$, and $2 \sum_\tau \mathrm{val}(\Pi_\tau)$ at inert $p > k$.

Problem 63.5.1. $12 \cdot 13 \cdot 14 \cdot 15 = 32760$ has $v_{13} = 1$. A product of u consecutive integers has $v_p \ge \lfloor u/p \rfloor$.

Problem 63.5.2. If $\mathrm{lv}_n(\nu) \le \rho$, then the H_i in Definition 63.3.2 may be taken to be monomials $\prod_\beta y(\mu_\beta z)^{e_{i,\beta}}$ with $\sum_\beta e_{i,\beta} \le |\nu|$.

For a modulus $\mathfrak{m}$ of E (Definition 58.2.1) with ideal (π_M) and $y_0 \in (\mathcal{O}_E/\mathfrak{m})^{12}$, let Th_{y_0} be the theta function (the th_φ of Definition 50.4.3; in this sheet th is the theta function of the curve) of the datum on $\mathrm{Hom}(E, E^{12})$ with the Gaussian at infinity, the indicator of y_0 at the places dividing $\mathfrak{m}$, and the vacuum at the other finite places. For a vector F on $\mathrm{Hom}(E, E^{12})$ write $\mathrm{Jet}(F)$ for its jet series at scale $\pi_M c_{\mathrm{jet}}$; for F with the Gaussian at infinity, as the datum of y_0, this is the Taylor series at 0, in the twelve copies, of $z \mapsto \mathrm{Th}_F(\pi_M c_{\mathrm{jet}} z)$, as in Definition 50.4.3. Write Jet_{y_0} for this series for the datum of y_0. Two facts are read from the proof. First, if $3e_{i,\beta} \le c_\beta$ for all $i \le 12$ and all β, the product over the twelve copies of the series $\prod_\beta y(\mu_\beta z)^{e_{i,\beta}} \,\mathrm{th}(\mu_\beta z)^{c_\beta}$ is η applied to $\mathrm{Jet}(F)$ for a vector F on $\mathrm{Hom}(E, E^{12})$ whose values at finite arguments are integral at η and whose theta function is entire and fixed by the signed translations along $\pi_M \mathfrak{a}_0^{12}$, with $\mathfrak{a}_0$ as in Definition 63.3.1. Second, for such F, finite Fourier inversion writes $M^{12f} F$ as a finite sum $\sum_j c_j \cdot$ (translate of the vacuum) such that $|\eta(c_j v_*)| \le 1$ for every value v_* of the vacuum at a finite argument, and each translate is a sum over y_0 of roots of unity times the datum of y_0. *Status*: both are read from the proof; for background on algebraic

theta functions compare Bannai and Kobayashi, Duke Math. J. 153 (2010). Here the integral vacuum of $\mathrm{Hom}(E, E^{12})$ is c_{vac}^{12} times the vacuum (Section 34.8). Also read from the proof: every conjugate under $\mathrm{Aut}(\mathbb{C})$ of a value of the integral vacuum of $\mathrm{Hom}(E, E^{12})$ at a finite argument has absolute value at most $\exp(v_0 q_1^2)^{12}$.

Problem 63.5.3. Let $y_1, \dots, y_N \in \mathbb{C}_p$, $c_1, \dots, c_N$ with $|c_i| \le 1$, and $|\sum_i c_i y_i| \ge \epsilon_0 > 0$. Then $|y_i| \ge \epsilon_0$ for some i. Fix a modulus as above and a value c_* of the integral vacuum of $\mathrm{Hom}(E, E^{12})$ at a finite argument (with $\mathrm{val}(0) = +\infty$). If $\mathrm{lv}_{12}(\lambda) \le \rho$ and $c_\beta \ge 3|\lambda|$ for all β, there are $\mathbf{h}$ with $\sum_i h_{\tau,i} = \lambda(\tau)$ and y_0 with $\mathrm{val}\,\eta\big(c_{\mathrm{vac}}^{12}[z^{\mathbf{h}}]\mathrm{Jet}_{y_0}\big) \le \rho + \mathrm{val}\,\eta(M^{12f} c_*)$.

Fix a nonzero value c_* of the integral vacuum of $\mathrm{Hom}(E, E^{12})$ at a finite argument, before any modulus or prime is chosen, and put $c_{\mathrm{att}} := M^{12f}\, c_* \prod_\tau \Pi_\tau^2$. If $\lambda(\tau) \le \kappa(\tau) + x^{42}$ for every τ, then $|\lambda| \le f(b + x^{42}) \le 2fb$, so $c_\beta \ge 6fb \ge 3|\lambda|$.

Problem 63.5.4. If $\lambda(\tau) \le \kappa(\tau) + x^{42}$ for every τ, then for every $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$\log|\sigma(c_{\mathrm{att}})| \le 12f \log M + 2fu \log(2b + u) + 12 v_0 q_1^2 \le \frac{fmb}{k}.$$

Problems 63.5.2, 63.5.3 and 63.5.4, with the bound on twelve copies (to which Problems 63.2.1, 63.3.3, 63.4.3 and 63.5.1 contribute), the tight frame and the two facts on theta functions, prove the following theorem.

Theorem 63.5.1. *Let $\lambda\colon \Phi \to \mathbb{N}$ satisfy $\kappa(\tau) \le \lambda(\tau) \le \kappa(\tau) + x^{42}$ for every τ. There is $c_{\mathrm{att}} \in \mathbb{C}$, $c_{\mathrm{att}} \ne 0$, with $|\sigma(c_{\mathrm{att}})| \le \exp(fmb/k)$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, such that: for every modulus $\mathfrak{m}$ of E with ideal (π_M), every prime p and every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_p$, there are a multi-index $\mathbf{h} = (h_{\tau,i})$ on the twelve copies with $\sum_i h_{\tau,i} = \lambda(\tau)$ for every τ, and a class $y_0 \in (\mathcal{O}_E/\mathfrak{m})^{12}$, with*

$$\mathrm{val}\,\eta\Big(c_{\mathrm{vac}}^{12}\, c_{\mathrm{att}}^{-1} \prod_{\tau \in \Phi} \lambda(\tau)! \cdot [z^{\mathbf{h}}]\,\mathrm{Jet}_{y_0}\Big) \le s_{\mathrm{fin},p}^{\mathrm{tot}}.$$

For fixed λ the same multiplier c_{att} works for all $\mathfrak{m}$, p and η; the witnesses $\mathbf{h}$, y_0 may depend on all three. Chapter 64 applies the theorem with λ the row λ_P of a critical pair main for j (Definitions 62.2.1 and 62.4.1), which satisfies the hypothesis by Problem 62.4.2.

Taken as given: the normal form, the isotropic scalings, the tight frame, the six-copy span and coupling bounds, the twelve-copy bound, the vacuum bound and the two facts on theta functions. The next sheet needs Theorem 63.5.1 and (1).

In the Lean formalization. In the Lean proof [GL] (readable edition, `BoundAttained.lean`), Theorem 63.5.1 is `Data.exists_rpow_neg_localPrimeSumTotal_le_norm_coeff`, stated with norms, Jet_{y_0} being `twelveTaylor` of `indicatorDatum` and $s_{\mathrm{fin},p}^{\mathrm{tot}}$ `Data.localPrimeSumTotal`. Definitions 63.3.2 and 63.4.1: `CMLattice.LeastValLE`, `Data.IsNearPair`. Normal form: `Data.exists_hasNormalForm_resolutionForms`. Isotropic scalings: `CMLattice.expSubst_sub_rescale_mem`, `CMLattice.prod_rescale_theta_mem`. Tight frame: `Data.exists_isTightFrame`. Coupling bound: `CMLattice.leastValLE_partnerDegree_of_isCo` Twelve copies: `Data.leastValLE_twelve`. The first fact on theta functions is `Data.exists_integral_isThetaFunction_twelveTaylor_eq`; the second is the Fourier expansion (`Space.exists_fourier_coefficients`) in the proof of `Data.exists_norm_le_norm_indicatorDatum`. The coefficient assertion in Problem 63.5.3 is `Data.exists_le_norm_coeff_indicatorDatum`. Further steps: `Data.mul_sum_weight_sub_localPrimeSumTotal`, `Parameters.cmCurve`,

`Parameters.periodFactor`, `Parameters.resolutionForms`, `CMLattice.leastValLE_of_unit_coefficients`, `Data.leastValLE_six_zero_of_inert`, `Data.isCouplingFamily_of_isNearPair`, `CMLattice.LeastValLE.exists_monomial`, `Parameters.norm_ringEquiv_intVacuum_le` and `Data.norm_map_mul_intVacuum_le`.

Hints

§63.1.

63.1.1: Problem 19.1.1, by machine.

63.1.2: Expand the exponential; for the integral series use Problem 19.4.2 and $\mu(6) \geq E_3(6)$.

63.1.3: By machine: the series of $F = \sigma^2 \wp$ through degree 52 from $\wp = \mathrm{sl}^{-2}$ and $(\log \sigma)'' = -\wp$ (Section 15.1), then $F(s+t)F(\zeta(s-t))$.

§63.2.

63.2.1: Compare term by term.

63.2.2: Euler's criterion.

§63.3.

63.3.1: Hensel's lemma at 13; for general p, Lagrange's theorem and the compactness of $\mathbb{Z}_p^4$.

63.3.2: Invert the one-variable product in $\mathfrak{o}[[y]]$, then substitute the forms.

63.3.3: Theorem 19.3.1 in total degree $|\nu|$ along μ_{β_0}, then specialize the six copies along an isotropic vector with first entry 1.

§63.4.

63.4.3: Start with (b): Φ_b lies in one class modulo q_0; compute d^e modulo q_0, using $d \equiv q_0/2 - 1$ and $4 \mid q_0$. For (a) at an inert prime, Problem 19.4.3 for Φ, $[-p]$, Φ_b, $L = l_0(s)$, the sparsity in Definition 33.3.1 and Problem 48.1.1.

§63.5.

63.5.2: As in Problem 19.1.1: truncate, and use the ultrametric inequality.

63.5.3: Separate the case $c_* = 0$. Otherwise apply the first part to the Fourier expansion multiplied by the vacuum value $c_*/c_{\mathrm{vac}}^{12}$, then to the expansion of a translate over the y_0; Problem 63.5.2 for the exponents.

63.5.4: Bound each factor of Π_τ by $2b + u$, and $|\sigma(c_*)|$ by the bound stated before Problem 63.5.3; for the last inequality use $M \leq b^{22}$, $k \leq r$ and the size conditions of Definition 33.3.1.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.