HECKE VALUES AS NORMS OF LIFTS

Prerequisites. Euler products of Hecke L-functions [Ne, Ch. VII, §8]. Chapter 23 (Sections 23.2, 23.3; Problems 23.2.3, 23.3.3, 23.3.4); Chapter 5 (heights, Section 5.3; Problems 5.3.1, 5.4.4); the data, B (Definition 33.4.1) and s_∞ of Chapter 33; from Chapter 34 the chosen places $\mathcal{M}$, $\mathcal{N}$, tame characters, υ, χ_M with residue characters ω, the vacuum scale c_{vac} and critical pairs; integrality at $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$ (Section 48.3); c_{jet} (Definition 50.4.3); Theorem 56.2.1; $\mathcal{H}_j$ (Definition 61.3.1). We write $|y|_\iota := |\iota(y)|_\ell$, $|\ell|_\ell = 1/\ell$.

For a regular family θ of m-fold twists and $1 \le j \le m$, Chapter 60 bounds the height of $Z_j(1)/Z_j(\theta_\iota)$, and Problem 61.3.2 expresses this quotient through the compression determinants and the quotient $\mathcal{H}_j(\vartheta)/\mathcal{H}_j(1)$ of products of Hecke values at $\vartheta = \theta_\iota^j$. How can a ratio of L-values be bounded at every $\iota \mid \ell$? For a critical pair with $z \ge 2$, the squared norm of a theta lift of a character from U(1), divided by $(\mathrm{Nm}\, Q_\xi)^{2z-1}$, is the partial Hecke value times a constant independent of the twist ξ; an integer multiple of it is integral away from $\mathrm{Nm}\, Q_\xi$; and for a main pair the untwisted squared norm has bounded height. We begin with a lift between two lines over $\mathbb{Z}[\mathrm{i}]$ and with the local factors in rank four over $\mathbb{F}_3$.

62.1. A Hecke value as a squared norm

In this section only, c_a, ϑ_0 and $L(\bar\upsilon_n, s)$ have the meanings of Sections 23.2 and 23.3. By Problem 23.2.3 the lift $\Theta_{(4)}$ between two lines is multiplication by $c_4/4$, and Problem 23.3.4 gives

$$\frac{c_4^2}{16} = 2100\, \vartheta_0^2\, (2\pi)^{-8}\, L(\bar\upsilon_8, 8).$$

Problem 62.1.1. To the precision shown, $c_4 \approx 0.14319133702$, $\vartheta_0 \approx 1.18034059902$, $L(\bar\upsilon_8, 8) \approx 1.06394325884$, and both sides of the display are ≈ 0.00128148493742.

This lift has rank one, which Theorem 62.4.1 excludes (it needs rank $2z \ge 4$); it shows only the shape of part (i), a squared norm equal to a fixed constant times a Hecke value. The size of the L-value itself is controlled by its Euler product, once the coefficients are unitary.

Problem 62.1.2. $L(\bar\upsilon_8, 8) = \sum_{\mathfrak{a}} (\bar\alpha/|\alpha|)^8\, \mathrm{Nm}\, \mathfrak{a}^{-4}$, the sum over the nonzero ideals $\mathfrak{a} = (\alpha)$ of $\mathbb{Z}[\mathrm{i}]$, with coefficients of absolute value 1. Hence $\log L(\bar\upsilon_8, 8) \le \log \zeta_{\mathbb{Q}(\mathrm{i})}(4)$; numerically $0.0620 \le 0.0680$.

In the proof the character is lifted to a definite space of rank n, with a finite datum on columns of length n at finitely many places. At each such place w, with residue field $\mathbb{F}_q$, three factors enter: the mean of the squared absolute value of the datum on $\mathbb{F}_q^n$, the index of the target level in the integral group, and $(1 - q^{-n})^{-1}$, restored from the denominator Euler product in the vacuum sum. At a *chosen* place the datum is the indicator of a fixed basis vector, and the level fixes that

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vector. At a place of the modulus Q_ξ it is the residue character of one coordinate, extended by zero, whose squared absolute value is the indicator of the complement of the coordinate hyperplane; the level preserves the hyperplane.

Problem 62.1.3. For $q = 3$, $n = 4$: one fixed vector is a proportion $1/81$ of $\mathbb{F}_3^4$, $\mathrm{GL}_4(\mathbb{F}_3)$ acts transitively on the 80 nonzero vectors, $(1 - 3^{-4})^{-1} = 81/80$, and the product is 1. In general $q^{-n}(q^n - 1)(1 - q^{-n})^{-1} = 1$.

Problem 62.1.4. For $q = 3$, $n = 4$: the vectors with nonzero first coordinate are a proportion $2/3$, $\mathrm{GL}_4(\mathbb{F}_3)$ acts transitively on the 40 hyperplanes, and $\frac{2}{3} \cdot 40 \cdot \frac{81}{80} = 27 = 3^3$. In general $\frac{q-1}{q}(1 + q + \cdots + q^{n-1})(1 - q^{-n})^{-1} = q^{n-1}$.

So the three factors give 1 at a chosen place and q^{n-1} at a place of the modulus: this is the power of $\mathrm{Nm}\, Q_\xi$ in Theorem 62.4.1. (The index of the source level of $\mathrm{U}(1)$ supplies a separate factor $q - 1$ at each chosen place.)

62.2. The lift of a character

Recall from Definition 34.10.1 that a critical pair $P = (\nu, z)$ has a character ν of weight $-\mathrm{wt}_P$, with $z \le \mathrm{wt}_P(\tau) \le 4b$, tame part trivial off $\mathcal{M}$, and $1 \le z \le t$; that $L^S(z, \mu)$ is the Euler product of the unitary Hecke character $y \mapsto \mu(y/\bar{y})$ of E, evaluated at chosen uniformizers, over the places outside $S \cup \bar{S}$, with its value at $z = 1$ given by the limit from the right whenever that limit exists in $\mathbb{C}$ (when $S \cup \bar{S}$ contains the ramified places, this is the partial Hecke L-function); and that for a tame character ξ and $\epsilon = \pm 1$

$$R_P(\xi, \epsilon) = \frac{|L^{\mathcal{M} \cup Q_\xi}(z, (\nu\xi^\epsilon)^{-1})|}{|L^{\mathcal{M}}(z, \nu^{-1})|}, \qquad R_P(\xi) = R_P(\xi, 1).$$

Definition 62.2.1. The *row* of a critical pair $P = (\nu, z)$ is $\lambda_P(\tau) := \mathrm{wt}_P(\tau) - z \ge 0$, for $\tau \in \Phi$. In this chapter $|\lambda_P| := \sum_{\tau \in \Phi} \lambda_P(\tau)$.

Definition 62.2.2. Let $P = (\nu, z)$ be a critical pair with $z \ge 2$ and ξ a tame character. Consider the theta lift of the constant function 1 on $\mathrm{U}(1)$ to the standard definite space of rank $2z$, for the character $\nu\xi$, with the following datum: at infinity the highest vector of the weight with one row of length $\lambda_P(\tau)$ at each τ; at the places of $\mathcal{M}$ the indicator of one fixed basis vector, at the places of Q_ξ the residue characters of ξ at one fixed coordinate, extended by zero (as in Section 62.1); the vacuum elsewhere. The *lift of the constant* $\phi_{P,\xi}$ is this lift times the index of the chosen level of $\mathrm{U}(1)$ and times $c_{\mathrm{vac}}^{2z}(\pi_M c_{\mathrm{jet}})^{|\lambda_P|}$.

The class sum of the theta kernel of this datum over $\mathrm{U}(1)$ at the chosen level equals the theta lift of 1 times the source level index; multiplying this class sum by $c_{\mathrm{vac}}^{2z}(\pi_M c_{\mathrm{jet}})^{|\lambda_P|}$ gives $\phi_{P,\xi}$. Its squared norm is computed by Theorem 56.2.1 for this sharp datum, which the proof checks (*Status*: the Rallis inner product formula [Ra, Li]). At infinity the Fock vector is paired with itself.

Problem 62.2.1. In the Fock space of Section 23.3, with variables $u_{\tau,1}, \ldots, u_{\tau,2z}$ at τ, the vector $\prod_\tau u_{\tau,1}^{\lambda_P(\tau)}$ has squared norm $\prod_\tau \lambda_P(\tau)!$.

Fix a critical pair $P = (\nu, z)$ with $z \ge 2$. The proof combines the formula of Theorem 56.2.1, the vacuum sum (the partial Hecke value divided by the product of $(1 - \varepsilon_w q_w^{-z})^{-1}$ over the finite places outside $\mathcal{M} \cup Q_\xi$ and its conjugates, with opposite signs ε_w at distinct conjugate places and $+1$ at a self-conjugate place), the local

factors of Problems 62.1.3 and 62.1.4 at the places of $\mathcal{M} \cup Q_\xi$, and Problem 62.2.1. It obtains a complex number $C_P \neq 0$, independent of ξ, with

$$\|\phi_{P,\xi}\|^2 = C_P \, L^{\mathcal{M} \cup Q_\xi}\big(z, (\nu\xi)^{-1}\big) \, (\mathrm{Nm}\, Q_\xi)^{2z-1}$$

for every tame character ξ. Here C_P is the squared modulus of the scale times the volumes of the two adelic quotients, divided by the full signed Euler product at z, times $\prod_{w \in \mathcal{M}} (q_w - 1)$, the hook number $\prod_\tau \lambda_P(\tau)!$ and the dimension of the weight. The lift is nonzero for every tame ξ.

Problem 62.2.2. Let χ be a unitary Hecke character of E, S a finite set of finite places, and (in this problem) $L^S(s, \chi)$ its L-function without the factors at S. For real $s \geq 2$, $L^S(s, \chi)$ is nonzero and $|\log |L^S(s, \chi)|| \leq [E : \mathbb{Q}] \log \zeta(2) = 2f \log \zeta(2) < 4f$.

The proof uses the bound $4f$.

62.3. Integrality, and the direction of the estimate

Definition 62.3.1. For a critical pair $P = (\nu, z)$ with $z \geq 2$ let $d_P := M^{4fz} (Md)^{|\lambda_P|}$, and for a tame character ξ let the *algebraic part* be

$$\mathrm{alg}_P(\xi) := \mathcal{N} \, d_P^2 \, \frac{\|\phi_{P,\xi}\|^2}{(\mathrm{Nm}\, Q_\xi)^{2z-1}}.$$

At primes $\ell \nmid \mathrm{Nm}\, Q_\xi$ the number d_P clears the denominators of the scaled theta kernel: M^{4fz} those of the finite datum, and Md per box of the row those of the derivatives. The proof shows that for every such ℓ and every ring homomorphism $\eta \colon \mathbb{C} \to \mathbb{C}_\ell$ the form $d_P \phi_{P,\xi}$ is integral at η (*Status*: read from the code). The squared norm is a class sum, $\|\phi\|^2 = \sum_c |\Gamma_c|^{-1} \langle \phi(g_c), \phi(g_c) \rangle$, over representatives g_c with finite stabilizers Γ_c.

Problem 62.3.1. Let I and T be finite sets, G an $I \times I$ hermitian matrix with entries in $\mathbb{Z}$, and $\eta \colon \mathbb{C} \to \mathbb{C}_\ell$ a ring homomorphism. For $c \in T$ let $a_c \in \mathbb{C}^I$, and let γ_c be a positive integer dividing a fixed integer $A \geq 1$. If $|\eta(a_{c,i})| \leq 1$ and $|\eta(\bar{a}_{c,i})| \leq 1$ for all c, i, then $|\eta(A \sum_c \gamma_c^{-1} \, a_c^* G a_c)| \leq 1$.

Problem 62.3.1 models two inputs read from the code. At any finite adelic point, the Gram pairing of two values of gauge at most one, the first at $\eta \circ$ (complex conjugation) and the second at η, is integral at η. On the standard definite space of rank at most $2n_m$, for every level contained in the stabilizer of the standard lattice, every class stabilizer has order dividing $\mathcal{N}$.

Problem 62.3.2. Let $P = (\nu, z)$ be a critical pair with $z \geq 2$ and ξ a tame character. Given the norm formula of Section 62.2, the integrality of $d_P \phi_{P,\xi}$, the Gram-pairing integrality and the divisibility of the stabilizer orders: $|\eta(\mathrm{alg}_P(\xi))|_\ell \leq 1$ for every prime $\ell \nmid \mathrm{Nm}\, Q_\xi$ and every $\eta \colon \mathbb{C} \to \mathbb{C}_\ell$; at $\xi = 1$ this holds at every prime; and $R_P(\xi) = \mathrm{alg}_P(\xi)/\mathrm{alg}_P(1) = \big(\|\phi_{P,\xi}\|^2/(\mathrm{Nm}\, Q_\xi)^{2z-1}\big)/\|\phi_{P,1}\|^2$.

Fix a critical pair $P = (\nu, z)$ with $z \geq 2$. For $1 \leq j \leq m$ and a regular family θ of m-fold twists (Definition 64.2.3), write $\vartheta_\iota := \theta_\iota^j$. The family to be estimated has terms $R_P(\vartheta_\iota)^{-1} = \mathrm{alg}_P(1)/\mathrm{alg}_P(\vartheta_\iota)$. The denominator is integral at ι, as $\ell \nmid \mathrm{Nm}\, Q_{\vartheta_\iota}$; its inverse need not be small there.

Problem 62.3.3. For $n \geq 1$, the number $y_n := (2 + \mathrm{i})^n$ is integral at every ι, and $|\sigma(y_n)| = 5^{n/2}$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$; yet $|y_n^{-1}|_\iota = 5^n$ for $\iota \mid 5$ with $|2 + \mathrm{i}|_\iota = 1/5$.

Thus an integral denominator can have a large inverse at the specified embedding, while Theorem 62.4.1(ii) gives only an upper bound for the absolute value of the denominator.

62.4. The size of the untwisted norm

Definition 62.4.1. A critical pair $P = (\nu, z)$ is *main for* j if $z \geq 10$, $m < z \leq r$, $1 \leq j \leq m$, the tame part of ν is ω^j, and $\kappa_j(\tau) + r - t \leq \mathrm{wt}_P(\tau) \leq \kappa_j(\tau)$ for every τ. It has *empty row* if $z \geq 2$ and $\mathrm{wt}_P(\tau) = z$ for every τ, or equivalently if $z \geq 2$ and $\lambda_P(\tau) = 0$ for every τ.

Problem 62.4.1. $s_\infty(y+1) \geq s_\infty(y)$ for every integer $y \geq 0$.

Problem 62.4.2. If P is main for j, then $\kappa(\tau) \leq \lambda_P(\tau) \leq \kappa(\tau) + x^{42}$ for every $\tau \in \Phi$.

The conjugates of the squared norm are read from the code. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing K and each integer $z \geq 2$ there is $c_{\sigma,z} \in \mathbb{R}$ such that every critical pair $P = (\nu, z)$ satisfies

$$\log\bigl|\sigma\bigl(\|\phi_{P,1}\|^2\bigr)\bigr| = c_{\sigma,z} + r_{\sigma,P} + \sum_{\tau\in\Phi} s_\infty(\lambda_P(\tau))$$

with a real $r_{\sigma,P} \geq -4f$, and $r_{\sigma,P} \leq 4f$ if P has empty row. Here $r_{\sigma,P}$ is the logarithm of the dimension of the transported weight (which is 1 for an empty row) plus $\log|L|$ for the inverse transported character at z away from $\mathcal{M}$, bounded by the $4f$ of Problem 62.2.2. The factors $M^{|\lambda_P|}$, $\Omega^{-|\lambda_P|}$ and $\prod_\tau \lambda_P(\tau)!$ come from $|\pi_M|^2 = M$, from conjugating the scaled jets, and from Problem 62.2.1. (*Status*: read from the code, which uses the conjugation of CM theta series and the Chowla–Selberg formula for the common area of the lattices $\sigma(u)\Lambda^\sigma$, with Λ^σ the conjugate lattice and u a number of absolute value 1 at every finite place; classical background: [Sh, Thm. 5.4], Theorem 20.3.1.) The proof also uses that for $2 \leq z \leq r$, with the same z as P, the pair $P_0 := (\upsilon^{-z}, z)$ is a critical pair with empty row and trivial tame part, and that $\mathcal{N}d_{P_0}^2 \leq e^{x^{33}}$.

Problem 62.4.3. For the data, $x^{33} + 8f \leq B$.

Problem 62.4.4. Let $P = (\nu, z)$ be main for j and $H := \mathrm{alg}_{P_0}(1)/\|\phi_{P,1}\|^2$. Then $\log|\sigma H| \leq B - \sum_\tau s_\infty(\kappa(\tau))$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $|H\|\phi_{P,1}\|^2|_\iota \leq 1$ for every ι. Hence the constant family $\iota \mapsto \|\phi_{P,1}\|^2$ has height at most $B - \sum_\tau s_\infty(\kappa(\tau))$.

Problem 62.3.2 proves (i) and (ii) of the following theorem, and Problem 62.4.4 proves (iii).

Theorem 62.4.1. *Let $P = (\nu, z)$ be a critical pair with $z \geq 2$.* (i) *For every tame character ξ, $R_P(\xi) = \bigl(\|\phi_{P,\xi}\|^2/(\mathrm{Nm}\, Q_\xi)^{2z-1}\bigr)/\|\phi_{P,1}\|^2$.* (ii) *For every tame character ξ, every prime $\ell \nmid \mathrm{Nm}\, Q_\xi$ and every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$, $|\eta(\mathrm{alg}_P(\xi))|_\ell \leq 1$.* (iii) *If P is main for j, the constant family $\iota \mapsto \|\phi_{P,1}\|^2$ has height at most $B - \sum_{\tau\in\Phi} s_\infty(\kappa(\tau))$.*

62.5. The Hecke product as a product of ratios

Recall from Definition 61.3.1 that $\mathcal{H}_j(\vartheta)$ is a product over $i \in \{\varnothing, +, -\}$ of $(\prod_{w\in S} q_w)^{z_i^2} \prod_{v=1}^{z_i} |L^S(v, (\nu_i \vartheta^{\epsilon_i})^{-1})|^2$, with $S = \mathcal{M} \cup Q_\vartheta$, lengths $z_\varnothing = (7m + k)/2$, $z_+ = m$, $z_- = m/2$, signs $\epsilon_\varnothing = \epsilon_+ = 1$, $\epsilon_- = -1$, and characters $\nu_\varnothing =$

$\mu_j^0 v^{j+(t-11m-1)/2}$, $\nu_\pm = v^{-z_\pm}\chi_M^{\pm j}$; each (ν_i, z_i) is a critical pair. At $v = 1$ we use that $L^S(1,\mu) \neq 0$ for an automorphic character μ of U(1) with nonzero weight and a tame part, for every finite set S of finite places of E (*Status*: classical [La, Ch. XV]).

Problem 62.5.1. Let $1 \le j \le m$ and let ϑ be a twist. (a) For $1 \le v \le z_i$, (ν_i, v) is a critical pair, and

$$\frac{\mathcal{H}_j(\vartheta)}{\mathcal{H}_j(1)} = (\mathrm{Nm}\, Q_\vartheta)^{z_\varnothing^2+z_+^2+z_-^2}\Big(\prod_i \prod_{v=1}^{z_i} R_{(\nu_i,v)}(\vartheta, \epsilon_i)\Big)^2.$$

(b) All these pairs are ramified; the $k/2$ pairs $(\nu_\varnothing, v)$ with $7m/2 < v \le z_\varnothing$ are moreover main for j, and the remaining indices number $5m$. (c) Formally for $m = 2$, $k = 4$: $(z_\varnothing, z_+, z_-) = (9, 2, 1)$, the exponent is 86, and the index range $7m/2 < v \le z_\varnothing$ has 2 positions, leaving 10 others.

At $\vartheta_\iota = \theta_\iota^j$ for a regular family, the inverse of $\mathcal{H}_j(\vartheta_\iota)/\mathcal{H}_j(1)$ has the factor $(\mathrm{Nm}\, Q_{\vartheta_\iota})^{-(z_\varnothing^2+z_+^2+z_-^2)}$, a unit at $\iota \mid \ell$. For the $5m$ remaining factors, regularity bounds the height of the inverse ratios by x_1 (Definition 64.2.2). For the $k/2$ main factors see Theorem 64.4.1.

We have proved Theorem 62.4.1 and the factorization of Problem 62.5.1. Taken as given: the norm formula of Section 62.2 (Theorem 56.2.1 for the sharp datum, the vacuum sum, the nonvanishing of the lift); the integrality of $d_P\phi_{P,\xi}$, of Gram pairings, and the divisibility of the stabilizer orders by $\mathcal{N}$; the formula of Section 62.4 for the conjugates of $\|\phi_{P,1}\|^2$, with the pair P_0 and the bound $\mathcal{N} d_{P_0}^2 \le e^{x^{33}}$; and $L^S(1,\mu) \neq 0$. Chapter 64 supplies the lower bound for $|\mathrm{alg}_P(\vartheta_\iota)|_\iota$ that Problem 62.3.3 shows is missing; it needs Theorem 62.4.1, Definitions 62.2.1–62.3.1 and Problems 62.4.2 and 62.5.1.

In the Lean formalization. In the Lean proof [GL] (`grh/src/`, namespace `New.` omitted; files in `New/OneLift/HeckeValues/` unless a path is given; auxiliary steps by name only): Theorem 62.4.1 is `Data.CriticalPair.ratio_eq_algPart_div` (`TwistedRatio.lean`), `Data.CriticalPair.integralAlgebraicPart_mem_integralAway` (`AlgebraicPartIntegrality.lean`) and `Data.heightLe_norm_trivialLift_sq` (`AlgebraicPartSize.lean`). Definitions 62.2.2 and 62.3.1 are `Data.CriticalPair.trivialLift` and `Data.CriticalPair.integralAlgebraicPart` (`CharacterLiftNorm.lean`), and Definition 62.4.1 is `Data.CriticalPair.IsMainPair` (`New/Determinant/Twist/FitTwists.lean`) and `Data.CriticalPair.HasEmptyRow` (`CharacterLiftNorm.lean`). The norm formula is derived from `LValueData.inner_trivialLift_self_eq_L` (`CharacterLiftNorm.lean`), which is stated for the class sum, without the scale $c_{\mathrm{vac}}^{2z}(\pi_M c_{\mathrm{jet}})^{|\lambda_P|}$; the integrality of $d_P\phi_{P,\xi}$ is `Data.CriticalPair.denomBound_smul_trivialLift_mem_integralForms`, and the displayed formula of Section 62.4 is `Data.CriticalPair.exists_log_norm_map_norm_sq_trivialLift_eq_add_sum_jetSize`. Auxiliary steps (a leading dot repeats the preceding namespace): `Data.CriticalPair.lValueData`, `.rowLengthSum`, `DescentPair.residueBlockDatum`, `LValueData.relIndex_descentResidueLevel`, `.stdLevelIndex_chosenLevel`,

`LValueData.trivialLift`, `.trivialLift_eq_smul_thetaLift`, `.isSharp_blockDatum`, `.inner_thetaLift_constForm_self`, `.integral_fockInner_realPairWeil`, `.hookNumber_eq_prod_factorial`, `Data.CriticalPair.exists_ne_zero_algPart_eq_mul_L`, `.L_inv_mul`, `.trivialLift_ne_zero`, `.denomBound`, `.weylGauge_denomBound_smul_pointKernel_le_one`, `LevelPlaces.LevelForm.inner_eq_stdClassSum`, `Data.CriticalPair.ratio_eq_integralAlgebraicPart_div`, `.norm_trivialLift_sq_eq_mul_inner`, `.exists_norm_map_inner_trivialLift_eq_mul`, `LValueData.map_trivialLift`; local factors, `LValueData.blockMeanSquare_mul_relIndex_mul_prod_eq_pow`; vacuum sum, `vacuumSum_eq_L_div_eulerProd`; Euler bound, `eulerProd_ne_zero_and_abs_log_norm_le`, specialized to $4f$ in `Data.exists_log_norm_inner_trivialLift_eq`; pairings and stabilizers, `gramForm_mem_embValRing` and `LevelPlaces.LevelForm.norm_nsmul_inner_le_one` and `Data.card_classAut_dvd_chosenGroupOrder`; CM, `CMLattice.complex_multiplication_theta` and `CMLattice.chowla_selberg`; P_0 and $e^{x^{33}}$, `Data.exists_hasEmptyRow` and `Data.CriticalPair.chosenGroupOrder_mul_denomBound_sq_le`; nonvanishing at 1, `AutChar.HasWeight.L_ne_zero`; the Hecke product, `Data.exchangePair`, `Data.CriticalPair.lowerPoint`, `Data.heckeValueProd_eq_mul_prod_ratio_sq`, `Data.isMainPair_lowerPoint_exchangePair_none`, `Data.isRamified_lowerPoint_exchangePair` and `Data.exists_heckeValueProd_eq_mul_prod_ratio`; the inverse ratios of main pairs, `Data.heightLe_inv_ratio`.

Hints

62.1.2: $\bar\alpha^8|\alpha|^{-16} = (\bar\alpha/|\alpha|)^8|\alpha|^{-8}$; compare with $\zeta_{\mathbb{Q}(\mathrm{i})}(4)$.
62.2.1: Problem 5.4.4; the norm factors over the variables.
62.2.2: Compare with $\zeta_E(s) \le \zeta(s)^{[E:\mathbb{Q}]}$.
62.3.3: Problem 5.3.1.
62.3.2: The Gram-pairing bound at each representative of the class sum, as in Problem 62.3.1; $\operatorname{Nm} Q_\xi$ is a unit at ℓ. For the ratio compare the norm formula at ξ and at 1.
62.4.1: Problem 33.4.3.
62.4.2: $z \le r$; $\kappa_j - \kappa = (j + \frac12)\Delta$ (Section 34.2); Definition 33.3.1.
62.4.3: $B = fxb\log^3 b/(mk)$ with $mk \le 6x^{11/2}$, and $b \ge x^{10^{12}}$.
62.4.4: Subtract the displayed formulas for P_0 and P (same z); then Problems 62.4.1, 62.4.2, 62.4.3, and 62.3.2 at $\xi = 1$. An automorphism not fixing K is complex conjugation composed with one that does.
62.5.1: (a) $\mathcal{M}$ and Q_ϑ are disjoint; divide each factor by its value at $\vartheta = 1$. (b) Problem 34.10.1, also for ω^{-j}. For the weights, μ_j^0 has weight $-\kappa_j$ and υ weight 1; use $r \ge x^5$, $m \le 2\sqrt{x}$ (Definition 33.3.1).

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