COMPRESSIONS AND FALSE LIFTS

Prerequisites. Linear algebra over $\mathbb{C}$: adjoints, orthogonal projections, determinants as products of eigenvalues. The fields $\bar{\mathbb{Q}}_\ell$ and $\mathbb{C}_\ell$, with $|\ell|_\ell = 1/\ell$. Inner products are conjugate-linear in the first variable. From Chapter 29: congruence modules, Problems 29.1.1 and 29.1.2. From Chapter 23: theta lifts between definite spaces and their norms (Theorem 23.3.1). From Chapter 5: isomorphisms $\iota \mid \ell$ and heights of families (Section 5.3); from Chapter 6, Theorem 6.2.1, which needs a height bound for $Z(1)/Z(\theta_\iota)$. For Section 61.3 only: the data (Chapter 33) and the spaces, forms, twists and single lifts of Chapter 34; the gauge and forms integral at a ring map η of Section 48.3; the exchange factors of Chapter 58; Chapters 56 and 57 for two facts recalled there; and Theorem 24.2.1 as a model of a Hecke relation.

This sheet serves Theorem 60.2.1 of Chapter 60, *The cost of one lift*: for a regular family (θ_ι) of m-fold twists (Definition 64.2.3), a bound for the height of $\iota \mapsto Z_j(1)/Z_j(\theta_\iota)$, where $Z_j(\theta_\iota)$ is the Gram determinant of the j-th single lift at the twist θ_ι^j. How can the ratio be bounded without evaluating either determinant? The Gram determinant at a twist, times the determinant of the compression of a second operator to the image of the lift, is a fixed nonzero number times a power of a ratio of Hecke values (Theorem 61.3.1). The compression determinant is bounded under $\mathrm{Aut}(\mathbb{C})$ at the trivial twist, and at ℓ at a twist prime to ℓ at which the projection is cleared. The second bound projects integral forms orthogonal to the *false lifts*: forms that satisfy the equations of a lift and are orthogonal to the lifts (Definition 61.3.2). We model this on lines in $\mathbb{C}^3$, then state the theorem of the proof.

61.1. Compressions

Let A and V be finite-dimensional complex inner product spaces and $u\colon A \to V$ an injective linear map, with image R. Write pr_R for the orthogonal projection of V onto R.

Definition 61.1.1. For $G \in \mathrm{End}\, V$, the *compression* of G to R is $\mathrm{pr}_R\, G|_R \in \mathrm{End}\, R$.

Problem 61.1.1. For every $G \in \mathrm{End}\, V$,

$$\det(u^*u)\, \det(\mathrm{pr}_R\, G|_R) = \det(u^*Gu).$$

Now let ϑ run over a set with an element 1, and suppose given for each ϑ an injective $u_\vartheta\colon A \to V_\vartheta$ with image R_ϑ, an operator $\Upsilon_\vartheta \in \mathrm{End}\, V_\vartheta$, and a complex number $\rho_\vartheta \neq 0$. In this section and the next, $Z(\vartheta) := \det(u_\vartheta^* u_\vartheta)$, $D_\vartheta := \det(\mathrm{pr}_{R_\vartheta}\, \Upsilon_\vartheta|_{R_\vartheta})$ and ρ_ϑ are local notations, models of the objects of Section 61.3.

Problem 61.1.2. Suppose that some $\Upsilon^\circ \in \mathrm{End}\, A$ has $\langle y, \Upsilon^\circ y\rangle > 0$ for every $y \neq 0$ and that $\langle u_\vartheta y', \Upsilon_\vartheta u_\vartheta y\rangle = \rho_\vartheta \langle y', \Upsilon^\circ y\rangle$ for all ϑ and $y', y \in A$. Then $\det \Upsilon^\circ \neq 0$, every D_ϑ is nonzero, and

$$Z(\vartheta)\, D_\vartheta = \det(\Upsilon^\circ)\, \rho_\vartheta^{\dim A} \qquad \text{for every } \vartheta.$$

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If moreover $\rho_1 = 1$, then $Z(1)/Z(\vartheta) = (D_\vartheta/D_1)\,\rho_\vartheta^{-\dim A}$.

So $Z(\vartheta)$ is traded for ρ_ϑ and D_ϑ. The simplest operators have rank one.

Problem 61.1.3. Let $A = \mathbb{C}$, $u_\vartheta(1) = v_\vartheta$, and $\Upsilon_\vartheta = g_\vartheta g_\vartheta^*$ with $g_\vartheta\colon \mathbb{C} \to V_\vartheta$, $g_\vartheta(1) = w_\vartheta$. Then $Z(\vartheta) = \|v_\vartheta\|^2$ and $D_\vartheta = |\langle w_\vartheta, v_\vartheta\rangle|^2/\|v_\vartheta\|^2$. If $\langle w_\vartheta, v_\vartheta\rangle = c_\vartheta^{-1}\mathrm{per}$ with numbers $c_\vartheta \neq 0$ and $\mathrm{per} \neq 0$, per independent of ϑ, the hypothesis of Problem 61.1.2 holds with $\rho_\vartheta = |c_1|^2/|c_\vartheta|^2$ and $\Upsilon^\circ = |\mathrm{per}|^2/|c_1|^2$.

The *toy* is the following pair of lines in $\mathbb{C}^3$, with the standard inner product, and the set $\{1, \vartheta\}$:

$$v_1 = (1,-1,0), \quad v_\vartheta = (1,1,-2), \qquad w_1 = (2,0,1), \quad w_\vartheta = (1,1,0),$$

with $u_1(z) = zv_1$, $u_\vartheta(z) = zv_\vartheta$, $\Upsilon_1 = w_1 w_1^*$ and $\Upsilon_\vartheta = w_\vartheta w_\vartheta^*$.

Problem 61.1.4. In the toy $Z(1) = 2$ and $Z(\vartheta) = 6$, $D_1 = 2$ and $D_\vartheta = 2/3$, and $Z(1)D_1 = Z(\vartheta)D_\vartheta = 4$.

Problem 61.1.5. Replace w_ϑ by $3w_\vartheta$. Then $D_\vartheta = 6$, the hypothesis of Problem 61.1.2 holds with $\Upsilon^\circ = 4$, $\rho_1 = 1$ and $\rho_\vartheta = 9$, and $(D_\vartheta/D_1)\,\rho_\vartheta^{-1} = 1/3 = Z(1)/Z(\vartheta)$.

Determinants are bounded through eigenvalues.

Problem 61.1.6. Let Y be a finite-dimensional complex vector space, $G \in \operatorname{End} Y$, $f_0\colon \mathbb{C} \to F_0$ a ring homomorphism into a field with an absolute value, and $c \in \mathbb{R}$. If $|f_0(\alpha)| \le c$ for every eigenvalue α of G, then $|f_0(\det G)| \le c^{\dim Y}$.

61.2. False lifts in $\mathbb{C}^3$

Where do bounds for D_ϑ come from? Let $V = \mathbb{C}^n$ with $\langle y, z\rangle = \sum \bar{y}_i z_i$. An automorphism σ of $\mathbb{C}$ acts on V coordinatewise; put $\langle y, z\rangle_\sigma := \langle \sigma y, \sigma z\rangle$, so that $\langle y, y\rangle_\sigma = \|\sigma y\|^2$. For $\iota \mid \ell$ let $\mathcal{L}_\iota := \{y \in V : |\iota(y_i)|_\ell \le 1 \text{ for all } i\}$, the vectors *integral at* ι.

In this section let $X \subseteq V$ be a subspace, the space cut out by the equations that every lift satisfies, and $R \subseteq X$ a subspace, the *lifts*. Put $F := X \cap R^\perp$, the *false lifts*: they satisfy the equations that define X but are orthogonal to the lifts. Let $\Upsilon \in \operatorname{End} V$ with $\Upsilon(R) \perp F$, and put $D := \det(\mathrm{pr}_R\, \Upsilon|_R)$. In Section 61.3, X is the intersection of the image of an idempotent with the kernel of a Hecke operator.

Problem 61.2.1. If $y \perp F$, then $\mathrm{pr}_R\, y = \mathrm{pr}_X\, y$.

Problem 61.2.2. Let $e, \mathbf{H} \in \operatorname{End} V$ with $e = e^* = e^2$, and $X = \operatorname{im} e \cap \ker \mathbf{H}$. Then $F^\perp = R + \ker e + \operatorname{im} \mathbf{H}^*$. In the toy, with e the identity and $\mathbf{H}$ the matrix with all entries 1, $X = \{y : y_1 + y_2 + y_3 = 0\}$ contains $R_1 = \mathbb{C}v_1$ and $R_\vartheta = \mathbb{C}v_\vartheta$; the false lifts are $F_1 = \mathbb{C}v_\vartheta$ and $F_\vartheta = \mathbb{C}v_1$, and $F_1^\perp = \mathbb{C}v_1 + \mathbb{C}(1,1,1)$.

The projection to R can fail to commute with σ. The next two problems give sufficient conditions for $\|\sigma(\mathrm{pr}_R\, y)\| \le \|\sigma y\|$ when $y \perp F$.

Problem 61.2.3. Let $K_0 \subset \mathbb{C}$ be a subfield, Galois over $\mathbb{Q}$ with abelian Galois group, such as $\mathbb{Q}$ or $\mathbb{Q}(\mathrm{i})$, and suppose that X is spanned by vectors with coordinates in K_0. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $y \in V$, $\sigma(\mathrm{pr}_X\, y) = \mathrm{pr}_{\sigma X}(\sigma y)$, where σX is spanned by the images under σ of such spanning vectors. Hence $\|\sigma(\mathrm{pr}_R\, y)\| \le \|\sigma y\|$ whenever $y \perp F$.

The next problem models, without rationality, the adjoint identities stated after Definition 61.3.2.

Problem 61.2.4. Let $e, \mathbf{H}$ and X be as in Problem 61.2.2, and suppose that for every $\sigma \in \mathrm{Aut}(\mathbb{C})$: $\langle ey, z\rangle_\sigma = \langle y, ez\rangle_\sigma$ for all y, z, and $\langle \mathbf{H}y, z\rangle_\sigma = 0$ whenever $\mathbf{H}z = 0$. Then $\operatorname{im}\mathbf{H}^* = \operatorname{im}\mathbf{H}$. For every $y \perp F$ there are $z \in R$ and $x \in V$ such that $y = z + x$ and $\langle x, z'\rangle_\sigma = 0$ for every $z' \in R$ and every $\sigma \in \mathrm{Aut}(\mathbb{C})$. For each such y, $\|\sigma(\mathrm{pr}_R\, y)\| \le \|\sigma y\|$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Problem 61.2.5. Suppose that $\|\sigma(\mathrm{pr}_R\, y)\| \le \|\sigma y\|$ for every σ and every $y \perp F$, and that $\|\sigma(\Upsilon y)\| \le \exp(c)\,\|\sigma y\|$ for all $y \in V$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then $|\sigma(D)| \le \exp(c \dim R)$ for all σ.

Problem 61.2.6. Suppose that $\Upsilon(\mathcal{L}_\iota) \subseteq \mathcal{L}_\iota$, and that $\Lambda \in \operatorname{End} R$ satisfies $\Lambda(\mathrm{pr}_R\, y) \in \mathcal{L}_\iota$ for every $y \in \mathcal{L}_\iota$ with $y \perp F$. Then $|\iota(\det\Lambda \cdot D)|_\ell \le 1$.

For X spanned by vectors in $\mathbb{Q}^n$, the least positive integer c with $c\,\mathrm{pr}_X(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$ is the exponent of the congruence module of X for $\mathbb{Q}^n = (X \cap \mathbb{Q}^n) \oplus (X^\perp \cap \mathbb{Q}^n)$ (Problem 29.1.2). We return to the toy, with $w_\vartheta = (1, 1, 0)$ and X as in Problem 61.2.2.

Problem 61.2.7. For the decomposition $\mathbb{Q}^3 = (X \cap \mathbb{Q}^3) \oplus \mathbb{Q}(1,1,1)$, the congruence module of X in $\mathbb{Z}^3$ is $\mathbb{Z}/3\mathbb{Z}$, and $3\,\mathrm{pr}_X(\mathbb{Z}^3) \subseteq \mathbb{Z}^3$. Hence, for $R = R_1$ and for $R = R_\vartheta$ and at every ι, $\Lambda = 3$ satisfies the hypothesis on Λ in Problem 61.2.6. For $R = R_1$ already $\Lambda = 1$ does.

Problem 61.2.8. In the toy, $\Upsilon_1(R_1) \perp F_1$ and $\Upsilon_\vartheta(R_\vartheta) \perp F_\vartheta$. The hypotheses of Problem 61.2.5 hold for Υ_1 with $\exp(c) = 5$, so $|\sigma(D_1)| \le 5$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$; those of Problem 61.2.6 hold for Υ_ϑ at $\iota \mid 3$ with $\Lambda = 3$, so $|D_\vartheta|_3 \le 3$, and equality holds.

So for rational X: if $c\,\mathrm{pr}_X(\mathbb{Z}^n) \subseteq \mathbb{Z}^n$ and Υ preserves $\mathcal{L}_\iota$, then $c^{\dim R} D$ is integral at ι; and under the hypotheses of Problem 61.2.5, $|\sigma(D)| \le \exp(c' \dim R)$, where $\exp(c')$ is the operator bound there. Neither bound uses $\det(u^* u)$.

61.3. The general statement

We use the data of Chapter 33 and the objects of Chapter 34: the chosen places $\mathcal{M}$ and their residue fields k_w, with $q_w := \#k_w$; the group $\bar{G}$ and the idempotent e_0; the good forms $H_g(W_j)$; twists ϑ, with modulus Q_ϑ and $\mathrm{Nm}\, Q_\vartheta$, and m-fold twists; the characters μ_j^0, υ and χ_M; the single lifts $u_{\xi,j}$ and $Z_j(\vartheta)$. Fix $1 \le j \le m$. For a twist ϑ, the single lift $u_{\vartheta,j}$ maps $H_g(W_j)$ to the space of algebraic forms on V_j with the weight, level and level character prescribed by the lift datum. Write $R_{\vartheta,j}$ for its image and $\mathrm{pr}_{\vartheta,j}$ for the orthogonal projection onto that image. We use two facts from Chapters 56 and 57: $u_{\vartheta,j}$ is injective for every tame character ϑ (Problem 56.4.2 with Theorem 56.4.1), and $\dim H_g(W_j) = d_T$ (Corollary 57.4.1).

For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and a form y on V_j, in this sheet $N_\sigma(y)$ is the class sum, weighted by the reciprocal orders of the stabilizers, of the squared Gram norms of the vectors $\sigma(y(g))$; $N_1(y) = \|y\|^2$. For a ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$, a form is *integral at* η if its value at every finite adelic point has gauge at most 1, the gauge of Section 48.3; integral at $\iota \mid \ell$ means integral at ι followed by $\bar{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$.

Definition 61.3.1. Put $z_\varnothing := (7m + k)/2$, $z_+ := m$, $z_- := m/2$, integers, as is $(t - 11m - 1)/2$, since $m = 2^e$ and $r = 3^{e'}$ (Definition 33.3.1); $\epsilon_\varnothing = \epsilon_+ = 1$ and

$\epsilon_- = -1$; $\nu_\varnothing := \mu_j^0\, \upsilon^{\,j+(t-11m-1)/2}$ and $\nu_\pm := \upsilon^{-z_\pm} \chi_M^{\pm j}$. The *Hecke value product* of lift j at a twist ϑ is

$$\begin{aligned}\mathcal{H}_j(\vartheta) &:= \prod_{i\in\{\varnothing,+,-\}} \mathrm{ex}_{z_i}\big(\mathcal{M}\cup Q_\vartheta,\, (\nu_i\vartheta^{\epsilon_i})^{-1}\big)\\ &= \prod_i \Big[\Big(\prod_{w\in\mathcal{M}\cup Q_\vartheta} q_w\Big)^{z_i^2} \prod_{n=1}^{z_i} \big|L^{\mathcal{M}\cup Q_\vartheta}\big(n, (\nu_i\vartheta^{\epsilon_i})^{-1}\big)\big|^2\Big],\end{aligned}$$

with the exchange factors of Definition 58.2.3 and the partial Hecke values $L^S(z,\nu)$ of Definition 34.10.1.

Let w be a finite place of E with $\bar w \neq w$ and $w, \bar w \notin \mathcal{M}\cup Q_\vartheta$. Let $\mathbf{H}_w$ be the Hecke polynomial of lift j at w, an operator on the forms on V_j made of the Hecke operators at w, evaluated at its test value at w, a number determined by the datum of the lift. The group $\bar G$ acts on these forms by right translation on the block of rank r of V_j, twisted by the tame part of the character of the lift, and e_0 acts through this action.

Definition 61.3.2. The *false lifts at* w are the forms in $\operatorname{im} e_0 \cap \ker \mathbf{H}_w$ that are orthogonal to $R_{\vartheta,j}$. A form is *orthogonal to the false lifts* if it is orthogonal to the false lifts at every such w.

For every such w, the image $R_{\vartheta,j}$ is contained in $\operatorname{im} e_0 \cap \ker \mathbf{H}_w$: this is the unramified Hecke relation of a theta lift, with the equivariance of the lift under $\bar G$; for its background see [Ra82], and Theorem 24.2.1 is a lattice-theta analogue. *Status*: we take the inclusion from the proof, as well as the estimate below. So $\operatorname{im} e_0 \cap \ker \mathbf{H}_w$ plays the part of X in Section 61.2, at each w. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ let $\mathfrak{b}_\sigma$ be the pairing on forms that is σ^{-1} of the class sum of the Gram pairings of the values after σ, so that $N_\sigma(y)$ is the real part of $\sigma(\mathfrak{b}_\sigma(y,y))$. The *projection estimate* is $N_\sigma(\mathrm{pr}_{\vartheta,j}\, y) \le N_\sigma(y)$ for every σ and every y orthogonal to the false lifts; its proof is that of Problem 61.2.4 at one place w, using that for each σ, e_0 is self-adjoint for $\mathfrak{b}_\sigma$ and $\mathbf{H}_w$ has an adjoint for $\mathfrak{b}_\sigma$ that commutes with $\mathbf{H}_w$.

Definition 61.3.3. A family of operators Υ_ϑ on the target space of $u_{\vartheta,j}$, one for every twist ϑ, is a *comparison operator* if: (1) the ratio $\mathcal{H}_j(\vartheta)/\mathcal{H}_j(1)$ is defined and nonzero for every twist ϑ; (2) some $\Upsilon^\circ \in \operatorname{End} H_g(W_j)$ has $\langle y, \Upsilon^\circ y\rangle > 0$ for $y \neq 0$ and $\langle u_{\vartheta,j} y', \Upsilon_\vartheta u_{\vartheta,j} y\rangle = (\mathcal{H}_j(\vartheta)/\mathcal{H}_j(1))\langle y', \Upsilon^\circ y\rangle$ for every twist ϑ and all $y', y \in H_g(W_j)$; (3) for every twist ϑ and $z \in R_{\vartheta,j}$, $\Upsilon_\vartheta z$ is orthogonal to the false lifts; (4) for every twist ϑ, every prime ℓ prime to $\operatorname{Nm} Q_\vartheta$, and every $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$, Υ_ϑ maps forms integral at η to forms integral at η; (5) $N_\sigma(\Upsilon_1 y) \le \exp(3fmb\log^2 b)^2\, N_\sigma(y)$ for every form y at the trivial twist and every $\sigma \in \mathrm{Aut}(\mathbb{C})$.

For a comparison operator put

$$D_\vartheta := \det\big(\mathrm{pr}_{\vartheta,j}\, \Upsilon_\vartheta|_{R_{\vartheta,j}}\big). \tag{1}$$

Definition 61.3.4. For $\iota \mid \ell$, a twist ϑ and a real number $\mathrm{cl}(\ell)$, say that $\mathrm{cl}(\ell)$ *clears the projection* at (ι,ϑ,j) if some $\Lambda \in \operatorname{End} R_{\vartheta,j}$ has $|\iota(\det\Lambda)|_\ell \ge e^{-(\dim R_{\vartheta,j})\, r\, \mathrm{cl}(\ell)}$ and $\Lambda(\mathrm{pr}_{\vartheta,j}\, y)$ is integral at ι for every y that is integral at ι and orthogonal to the false lifts.

Every class of the finite adelic group modulo the rational points and the level has a representative whose components above ℓ are integral, by weak approximation.

Problem 61.3.1. Let $y \neq 0$ be a form in the target space of $u_{\vartheta,j}$, let $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$, and fix class representatives integral above ℓ. At $g = g_0 s g_1$, with g_0 rational, s a representative and g_1 in the level, the gauge of $y(g)$ is the largest absolute value of a coordinate of $\eta(y(s))$. Hence there is $c \in \mathbb{C}^\times$ such that cy is integral at η and αcy is integral at η only if $|\eta(\alpha)|_\ell \leq 1$.

Problems 61.1.1, 61.1.2, 61.1.6, 61.2.5, 61.2.6 and 61.3.1, carried over to forms with the projection estimate, the facts that $N_\sigma(y) > 0$ for $y \neq 0$ and $N_\sigma(ay) = |\sigma(a)|^2 N_\sigma(y)$, the injectivity of $u_{\vartheta,j}$ and $\dim H_g(W_j) = d_T$ (so that $\dim R_{\vartheta,j} = d_T$), prove the following equivalent form of the three results identified in the Lean remark below.

Theorem 61.3.1. *Let $1 \leq j \leq m$ and let (Υ_ϑ) be a comparison operator.* (a) *The number $\det(u^*_{\vartheta,j} u_{\vartheta,j})\, D_\vartheta\, (\mathcal{H}_j(\vartheta)/\mathcal{H}_j(1))^{-d_T}$ is nonzero and does not depend on the twist ϑ.* (b) *For every twist ϑ, every $\iota \mid \ell$ with ℓ prime to $\operatorname{Nm} Q_\vartheta$, and every real $\operatorname{cl}(\ell)$ that clears the projection at (ι, ϑ, j), $|\iota(D_\vartheta)|_\ell \leq e^{d_T r \operatorname{cl}(\ell)}$.* (c) *$|\sigma(D_1)| \leq e^{3 d_T f m b \log^2 b}$ for every $\sigma \in \operatorname{Aut}(\mathbb{C})$.*

In particular every D_ϑ is nonzero. For an m-fold twist ϑ the power ϑ^j is a twist (Section 34.6), and $1^j = 1$.

Problem 61.3.2. For every comparison operator and every m-fold twist ϑ,

$$\frac{Z_j(1)}{Z_j(\vartheta)} = \frac{D_{\vartheta^j}}{D_1} \Big(\frac{\mathcal{H}_j(\vartheta^j)}{\mathcal{H}_j(1)} \Big)^{-d_T}.$$

Chapter 60 combines Problem 61.3.2 and Theorem 61.3.1(b), (c) with the comparison operator of Theorem 60.1.1 and the height bound of Problem 60.2.2 to prove Theorem 60.2.1. Taken as given here: the injectivity of the single lifts, $\dim H_g(W_j) = d_T$, the inclusion of the lifts in $\operatorname{im} e_0 \cap \ker \mathbf{H}_w$, and the projection estimate.

In the Lean formalization. In the Lean proof [GL], in `grh/src/New/OneLift/LiftVolumeHeight.lean` unless said otherwise: Theorem 61.3.1 is `Data.IsComparisonOperator.exists_normDet_sq_mul_det_eq` (a), `.norm_padic_det_liftCompression_le` (b) and `.norm_map_det_liftCompression_le` (c), stated with $\dim H_g(W_j)$ in place of d_T and with a named constant in (a); comparison operators are `Data.IsComparisonOperator`; $\mathcal{H}_j$ is `Data.heckeValueProd` (`New/OneLift/ComparisonOperator/ThreeExchanges.lean`); orthogonality to the false lifts is `DescentPair.IsOrthogonalToFalseLifts` (`New/OneLift/TestedForms.lean`); clearing is `Data.ClearsProjection` (`New/Determinant/Twist/FitTwists.lean`); the inclusion of the lifts in $\operatorname{im} e_0 \cap \ker \mathbf{H}_w$ is `Data.range_goodLift_le_range_inf_ker`, the projection estimate is `Data.normSqAt_starProjection_range_goodLift_le`, and the scaling conclusion of Problem 61.3.1 is the specialization of `DescentPair.exists_smul_mem_integralForms_maximal` to the datum of lift j, whose proof contains the gauge calculation.

Hints

§61.1.

61.1.1: Since u^* kills $R^\perp$, $u^*Gu = u^* \operatorname{pr}_R Gu$. Write u as an isomorphism $A \to R$ followed by the inclusion.
61.1.2: Problem 61.1.1 with $G = \Upsilon_\vartheta$; the hypothesis says $u_\vartheta^* \Upsilon_\vartheta u_\vartheta = \rho_\vartheta \Upsilon^\circ$.
61.1.3: $g_\vartheta^* v = \langle w_\vartheta, v \rangle$.
61.1.4: Problem 61.1.3.
61.1.6: Factor the characteristic polynomial over $\mathbb{C}$.

§61.2.

61.2.1: $X = R \oplus F$, orthogonally.
61.2.2: Take orthogonal complements in the definition of F.
61.2.3: $\operatorname{pr}_X = \mathsf{B}(\mathsf{B}^*\mathsf{B})^{-1}\mathsf{B}^*$ for a matrix B whose columns form a basis of X and have coordinates in K_0; on K_0, complex conjugation commutes with σ.
61.2.4: At $\sigma = 1$, $\ker \mathbf{H} \subseteq (\operatorname{im} \mathbf{H})^\perp = \ker \mathbf{H}^*$. Decompose y by Problem 61.2.2, and use $R \subseteq \operatorname{im} e \cap \ker \mathbf{H}$.
61.2.5: Take an eigenvector z of $\operatorname{pr}_R \Upsilon|_R$ and apply the hypothesis to Υz; then Problem 61.1.6.
61.2.6: Scale an eigenvector of $\Lambda \operatorname{pr}_R \Upsilon|_R$ to be integral at ι with a coordinate of absolute value 1.
61.2.7: Problem 29.1.1: the index of $(\mathbb{Z}^3 \cap X) \oplus \mathbb{Z}(1,1,1)$ in $\mathbb{Z}^3$; then Problems 61.2.1 and 29.1.2. For R_1, write $\operatorname{pr}_{R_1} y$ for $y \perp v_\vartheta$.
61.2.8: $\|w_1\|^2 = 5$, and w_1 has rational coordinates; for equality, Problem 61.1.4.

§61.3.

61.3.1: Use the transformation law at $g_0 s g_1$, whose level character has finite order, and compare the coordinates at all the representatives.
61.3.2: Theorem 61.3.1(a) at ϑ^j and at 1.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[Ra82] S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104** (1982), 469–515.