SQUARES AND THE TRIANGLE

Prerequisites. Inner products are conjugate-linear in the first argument. From Chapter 56: Theorem 56.4.1, the twist factor c_Q (Definition 56.3.1), the Rallis operators $\mathcal{R}_j(\xi; V)$ (Definition 56.3.2) and Problem 56.3.2. From Chapter 58: moduli $\mathfrak{m}$ and $S_{\mathfrak{m}}$ (Definition 58.2.1), the exchange factor $\mathrm{ex}_h(S, \nu)$ (Definition 58.2.3), Theorem 58.5.1 and Problem 58.5.5. From Chapter 34: $\mathcal{M}$, k_w, q_w, the spaces Y_j (rank $r + jt$), $H(V) \supseteq H_g(V)$, T, m-fold twists ϑ with tame part (ψ_w) and set Q_ϑ, the tame part (ω_w) of χ_M, of order q_2, the characters μ_j^0, the lower chain Θ_ϑ (Section 34.9), and $L^S(z, \nu)$ and critical pairs (Definition 34.10.1). Chapter 33 supplies $t = 5r$ and b, and Section 34.2 defines $\Delta = 2r^8$. Chapter 8 is motivation.

Theorem 55.4.1 factors $Z(\vartheta)/Z(1)$. Its input on the chain is the Gram identity of Theorem 59.4.1 below, proved in this sheet from three statements read from the proof: the realization of a triangle of lifts, the inner product formula on its first edges, and the comparison at the last level. By Theorem 56.4.1 the Gram operator of the single lift on $H(W_j)$ is $a_j c_{Q_\xi} \mathcal{R}_j(\xi; W_j)$. Why is the Gram operator of the chain on T a scalar multiple of the product of the m Rallis operators, and how do ratios of L-values enter that scalar?

59.1. Why a triangle

Problem 59.1.1. With $t = 5r$: $2r < r + t$, but $2(r + (j-1)t) > r + jt$ for every $j \geq 2$. So the condition $2n^\flat < n^\sharp$ of the inner product formula (Theorem 56.2.1) holds for the first lift $Y_0 \to Y_1$ of the chain and for no later lift $Y_{j-1} \to Y_j$.

The following identity would suffice if the Gram operators could be moved through the preceding lifts.

Problem 59.1.2. Let $\theta_j \colon \mathcal{V}_{j-1} \to \mathcal{V}_j$ $(1 \leq j \leq m)$ be linear maps of complex inner product spaces that have adjoints, and $\Theta_j := \theta_j \circ \cdots \circ \theta_1$. Suppose that $\theta_1^* \theta_1 = a_1 \mathcal{E}_1$ and that $\theta_j^* \theta_j \, \Theta_{j-1} = a_j \, \Theta_{j-1} \mathcal{E}_j$ for $2 \leq j \leq m$, with scalars a_j and endomorphisms $\mathcal{E}_j$ of $\mathcal{V}_0$. Then $\Theta_m^* \Theta_m = a_1 \cdots a_m \, \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_m$.

For $2 \leq j \leq m$ the ranks of the chain are outside the range of the inner product formula (Problem 59.1.1). The proof applies the formula only on the first edge of each row of a triangle of lifts, and relates the resulting Gram forms by two square laws.

59.2. The triangle and its two square laws

In this sheet a *triangle of size* m consists of inner product spaces $H_{i,n}$ $(i, n \geq 0$, $i + n \leq m)$; a space $H_\circ$ with linear isometric embeddings $H_\circ \to H_{i,0}$, through which we regard $H_\circ$ as a subspace of each $H_{i,0}$; linear maps $\lambda_{i,n} \colon H_{i,n} \to H_{i,n+1}$ (lifts) and $\delta_{i,n} \colon H_{i,n+1} \to H_{i+1,n}$ (descents) for $i + n < m$; and positive integer weights $g_0, g_1, \ldots$. The *weighted rows* are $\lambda_i^{(0)} := 1$ and $\lambda_i^{(n+1)} := g_n \lambda_{i,n} \lambda_i^{(n)} \colon H_{i,0} \to$

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$H_{i,n+1}$. The *square* (i,n), for $i+n+2\le m$, has corners $H_{i,n+1}$, $H_{i,n+2}$, $H_{i+1,n}$, $H_{i+1,n+1}$. We ask:

(F) for $i<m$, $x\in H_\circ$ and $v\in H_{i,1}$: $\langle v,\lambda_{i,0}x\rangle=\langle\delta_{i,0}v,x\rangle$; and $\delta_{i,0}\lambda_{i,0}$ maps $H_\circ\subseteq H_{i,0}$ injectively into $H_\circ\subseteq H_{i+1,0}$;

(D) the *direct law* at each square, with a scalar $c_{i,n}$: $g_n\lambda_{i+1,n}\delta_{i,n}=(g_{n+2}/\bar c_{i,n})\,\delta_{i,n+1}\lambda_{i,n+1}$ on $H_{i,n+1}$, where a quotient by 0 means 0;

(A) the *adjoint law* at each square, with a scalar $c^*_{i,n}$: every $v\in H_{i,n+2}$ has an $x\in H_{i,n+1}$ with $\langle\lambda_{i,n+1}y,v\rangle=\langle y,x\rangle$ for all y, and $\langle\delta_{i,n}x,y'\rangle=c^*_{i,n}\langle\delta_{i,n+1}v,\lambda_{i+1,n}y'\rangle$ for all $y'\in H_{i+1,n}$.

Problem 59.2.1. Let $m=2$, every $H_{i,n}=H_\circ=\mathbb{C}$ with the usual inner product and identity embeddings, $(g_0,g_1,g_2)=(2,3,5)$, and let the maps be multiplication by $\lambda_{0,0}=2$, $\lambda_{1,0}=3$, $\lambda_{0,1}=4$, $\delta_{0,0}=2$, $\delta_{1,0}=3$, $\delta_{0,1}=1$. Then (F) holds; the only square is $(0,0)$, its direct law holds exactly with $c_{0,0}=5/3$, and its adjoint law exactly with $c^*_{0,0}=8/3$.

Problem 59.2.2. In Problem 59.2.1, $\lambda_0^{(2)}=g_1\lambda_{0,1}g_0\lambda_{0,0}=48$, and $48^2=2304=c_{0,0}c^*_{0,0}\,(g_0g_1^2/g_2)\,(g_0\lambda_{0,0}^2)(g_0\lambda_{1,0}^2)$.

In Problems 59.2.3–59.2.6 the triangle is arbitrary and satisfies (F), (D), (A). For $i+n<m$ put $c_i^{[n]}:=c_{i,0}c_{i,1}\cdots c_{i,n-1}$ and $c_i^{*[n]}:=c^*_{i,0}\cdots c^*_{i,n-1}$ (empty products are 1); Problems 59.2.3, 59.2.5 and 59.2.6 assume $i+n<m$.

Problem 59.2.3. $\lambda_{i+1}^{(n)}\delta_{i,0}\lambda_{i,0}=\dfrac{g_{n+1}}{g_1g_0\,\overline{c_i^{[n]}}}\,\delta_{i,n}\lambda_i^{(n+1)}$ on $H_{i,0}$.

Problem 59.2.4. For $i+n\le m$, $\lambda_i^{(n)}$ is injective on $H_\circ$. If $H_\circ\ne0$ and $i+n<m$, then $c_i^{[n]}\ne0$.

Problem 59.2.5. For $x\in H_\circ$ and $v\in H_{i,n+1}$: $\langle v,\lambda_{i,n}\lambda_i^{(n)}x\rangle=c_i^{*[n]}\,\langle\delta_{i,n}v,\lambda_{i+1}^{(n)}x\rangle$.

Problem 59.2.6. For $x,y\in H_\circ$ and $x':=\delta_{i,0}\lambda_{i,0}x\in H_\circ$: $\langle\lambda_i^{(n+1)}x,\lambda_i^{(n+1)}y\rangle=c_i^{[n]}c_i^{*[n]}\,\dfrac{g_0g_1g_n}{g_{n+1}}\,\langle\lambda_{i+1}^{(n)}x',\lambda_{i+1}^{(n)}y\rangle$.

In the chain, $H_{i,n}$ consists of forms on a Hermitian space of rank $r+nt$ whose parameter has n blocks of length t, labelled $i+1,\dots,i+n$. In row 0 these spaces are $Y_0,\dots,Y_m$; the first space of row i is Y_0 with its form multiplied by $(-1)^i$. On labels, the lift $\lambda_{i,n}$ adds block $i+n+1$ and the descent $\delta_{i,n}$ removes block $i+1$. So the square (i,n) removes block $i+1$ and adds block $i+n+2$; we say it carries the pair $(i+1,i+n+2)$.

Problem 59.2.7. As (i,n) runs over the squares of a triangle of size m, the pair $(i+1,i+n+2)$ runs over the pairs $1\le i'<j'\le m$, each once; there are $m(m-1)/2$ squares. For $m=3$ the squares $(0,0),(0,1),(1,0)$ carry $(1,2),(1,3),(2,3)$.

59.3. The triangle of the chain and its scalars

For an m-fold twist ϑ let $\mathfrak m:=(\pi_M)\prod_{w\in Q_\vartheta}w$, so $S_{\mathfrak m}=\mathcal M\cup Q_\vartheta$, and in this sheet $\mathfrak R:=\mathcal O_E/\mathfrak m$, both depending on ϑ; $\mathfrak R\cong\prod_{w\in\mathcal M\cup Q_\vartheta}k_w$, as p_0,p_1 split completely in E and $\operatorname{Nm}Q_\vartheta$ is prime to M. In the triangle of the chain, $H_\circ=H(Y_0)$, the node $H_{i,n}$ is a space of forms on a space of rank $r+nt$, row 0 runs through $Y_0,\dots,Y_m$, the lift $\lambda_{i,n}$ is a theta lift built from the character $\mu^0_{i+n+1}\vartheta^{i+n+1}$ (times a power of

the splitting character υ, depending on the ranks) with the indicator of $\binom{1}{0}$ modulo $\mathfrak{m}$ as finite datum, and $g_n(\vartheta) := \#\mathrm{GL}_{r+nt}(\mathfrak{R})$. The descents are transposed theta lifts; in each row the proof chooses their polynomials at infinity and the finite datum of $\delta_{i,0}$ so that $\delta_{i,0}$ is adjoint to $\lambda_{i,0}$ on $H_\circ$. For a function p on matrices η of a fixed shape over $\mathfrak{R}$, the *next datum* is the function on the matrices with one more block of rows and of columns, of size t, that equals $p(-\eta)$, divided by the number of possible blocks η', at $\left(\begin{smallmatrix}\eta & \eta'\\ 0 & 1\end{smallmatrix}\right)$, and 0 at all other matrices; the finite datum of $\delta_{i,n+1}$ is the next datum of that of $\delta_{i,n}$.

Realization of the triangle. The proof shows (F), and (D), (A) at each square with functions $c_{i,n}$, $c^*_{i,n}$ of the m-fold twist, as follows. Theorem 58.5.1, on the sum of the two middle Hermitian spaces, gives a finite square law for the two composites around the square on point data; a finite summation turns it into (D) for a datum and its next datum, with $|c_{i,n}(\vartheta)|^2 = C_{i,n}\,\mathrm{ex}_t(\mathcal{M}\cup Q_\vartheta, \nu_{i,n}(\vartheta))$, $C_{i,n}\in\mathbb{R}$ independent of ϑ and $\nu_{i,n}(\vartheta)$ as in Problem 59.3.2. The proof also provides the adjoint vectors in (A), with $c^*_{i,n}\doteq 1$ in the sense of Definition 59.3.1; one of its factors is the scalar of the reversed square, which combines an exchange with partners of equal rank with the inverse of the Fourier constant of Problem 59.3.5. *Status*: read from the proof. For joint harmonics see [Li90, Ho89], for the seesaw principle [Ku84].

Definition 59.3.1. For functions c from m-fold twists to $\mathbb{C}$ and f to $\mathbb{R}$, $c\doteq f$ means: there are $C'\in\mathbb{R}$ (possibly 0) and $e'\in\mathbb{Z}$ with $|c(\vartheta)|^2 = C'(\mathrm{Nm}\,Q_\vartheta)^{e'} f(\vartheta)^2$ for every m-fold twist ϑ.

The relation $c\doteq f$ allows c to vanish identically; if c is nonzero at one twist, its zeros are exactly those of f.

Definition 59.3.2. For $1\le i<j\le m$, $\nu := \mu^0_j/\mu^0_i$ and an m-fold twist ϑ,

$$\rho_{ij}(\vartheta) := \prod_{z=1}^{t} \frac{|L^{\mathcal{M}\cup Q_\vartheta}\big(z,(\nu\vartheta^{j-i})^{-1}\big)|}{|L^{\mathcal{M}}(z,\nu^{-1})|},$$

a quotient by 0 being 0.

The values $L^S(z,\nu)$ in Definition 59.3.2 and in the exchange factors below are finite and nonzero, by the nonvanishing stated in Chapter 58, Section 58.2: the characters have weight $\pm(j-i)\Delta\neq 0$ and a tame part (Problems 59.3.2 and 59.4.6).

Problem 59.3.1. In the square (i,n) the Hermitian spaces underlying the middle corners $H_{i,n+1}$, $H_{i+1,n+1}$ have rank $r+(n+1)t$ each, and those underlying the partners $H_{i,n+2}$, $H_{i+1,n}$ have ranks $r+(n+2)t$ and $r+nt$. So in Theorem 58.5.1, with V^0 the orthogonal sum of the two middle Hermitian spaces, $n_0 = 2(r+(n+1)t)$ and $h=t$. In the reversed square the partners are the spaces underlying $H_{i,n+1}$, $H_{i+1,n+1}$, of equal rank, so $h=0$ and $\mathrm{ex}_0=1$.

For the data, $8m\le 16\sqrt{x} < x^{11}\le T(s)\le q_1 d$ (Definition 33.3.1, Problem 33.3.3) and $q_1 d \mid q_2$ (Chapter 33).

Problem 59.3.2. Let $\nu_{i,n}(\vartheta) := \big((\mu^0_{i+n+2}/\mu^0_{i+1})\vartheta^{n+1}\big)^{-1}$. It has weight $(n+1)\Delta$; moreover $(n+1)\Delta\ge r+(n+1)t$ and $n_0\le 2(n+1)\Delta$. Its tame part is $\omega_w^{-(n+1)}$ at $w\in\mathcal{M}$, $\psi_w^{-(n+1)}$ at $w\in Q_\vartheta$, trivial elsewhere, and nontrivial at every $w\in\mathcal{M}\cup Q_\vartheta$.

Problem 59.3.3. If $c\doteq f$ and $c'\doteq f'$, then $cc'\doteq ff'$. For every square, $c_{i,n}\doteq \rho_{i+1,\,i+n+2}$, with exponent $e'=t^2$, and $c^*_{i,n}\doteq 1$.

The powers of $\#\mathfrak{R}$ in the adjoint law come from finite Fourier transforms $\hat{F}(\mathbf{v}') := \sum_{\mathbf{v}} F(\mathbf{v})\,\mathbf{e}(\mathrm{tr}(\mathbf{v}'\mathbf{v}))$, with $\mathbf{e}$ a primitive additive character of $\mathfrak{R}$. For $\mathfrak{R} = \prod_w k_w$, with primitive characters $\mathbf{e}$ (additive) and χ (multiplicative, 0 on nonunits: every $a \neq 0$ has some $\chi(1+ca) \notin \{0,1\}$) there is $G \in \mathbb{C}$ with $|G|^2 = \#\mathfrak{R}^{t^2}$ such that every function F on $t \times t$ matrices with $F(k\mathbf{v}) = \chi(\det k)F(\mathbf{v})$ for invertible k has $\hat{F}(\mathbf{v}') = G\,\chi^{-1}(\det \mathbf{v}')\,F(1)$, with χ^{-1} also extended by 0. *Status*: for the Gauss sum of a determinant character over a finite field see [Ko63]; for $\mathfrak{R} = \prod_w k_w$ such an F vanishes off the invertible matrices and is a product over w, so the identity follows from the case of fields. At a node of rank $r + nt$ the Fourier identity for the conjugated data has constant $G \cdot \#\mathfrak{R}^{(r+nt)t}$: G from a corner block of size t, and $\#\mathfrak{R}^{(r+nt)t}$ from the sum over a free block of size $(r + nt) \times t$; the reversed square uses its inverse (in the proof of (A)).

Problem 59.3.4. Take the added block of size 1. Over $\mathfrak{R} = \mathbb{F}_3$, with p the indicator of $\{1\}$ on 1×1 matrices, the next datum is $1/3$ at the three matrices $\left(\begin{smallmatrix} -1 & \eta' \\ 0 & 1 \end{smallmatrix}\right)$ ($\eta' \in \mathbb{F}_3$) and 0 elsewhere. With $\mathbf{e}(x) = e^{2\pi i x/3}$, its Fourier transform at $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $1_{C=0}\,\mathbf{e}(-A + D)$.

Problem 59.3.5. For $\mathbf{e}$ as in Problem 59.3.4 and the nontrivial character χ of $\mathbb{F}_3^\times$, the Gauss sum $\sum_{x \in \mathbb{F}_3^\times} \chi(x)\,\mathbf{e}(x)$ is $i\sqrt{3}$. In general $|G \cdot \#\mathfrak{R}^{(r+nt)t}|^2 = \#\mathfrak{R}^{t(t+2(r+nt))}$; formally, for $\mathfrak{R} = \mathbb{F}_3$ and $r + nt = t = 1$, this is $|3i\sqrt{3}|^2 = 27$.

Problem 59.3.6. Over $\mathbb{F}_q$, the invertible block upper triangular matrices with diagonal blocks of sizes $r, t, \ldots, t$ (j blocks t) form a group of order $\#\mathrm{GL}_r(\mathbb{F}_q)\,\#\mathrm{GL}_t(\mathbb{F}_q)^j\,q^{jrt+t^2 j(j-1)/2}$. For blocks $(1,2)$ over $\mathbb{F}_2$ the order is 24; for $r = 1$, $t = 5$ the exponent is $5, 35, 90$ for $j = 1, 2, 3$.

59.4. The Gram identity of the chain

Definition 59.4.1. For an m-fold twist ϑ, let $\mathcal{R}_\vartheta$ be the endomorphism of T

$$\mathcal{R}_\vartheta := \mathcal{R}_1(\vartheta; Y_0)\,\mathcal{R}_2(\vartheta^2; Y_0) \cdots \mathcal{R}_m(\vartheta^m; Y_0).$$

Each factor preserves T (Corollary 57.4.1). In this sheet $c_{\mathrm{mat}}(\vartheta) := \#\mathfrak{R}^{r(r+t)}$, the number of $(r + t) \times r$ matrices over $\mathfrak{R}$. The inner product formula on the first lift of each row (Problem 59.1.1), with the point datum, gives: for $i < m$ there is $C_i \in \mathbb{C}$, independent of ϑ, with

$$g_0\,\langle \lambda_{i,0}x, \lambda_{i,0}y\rangle = \frac{C_i}{c_{\mathrm{mat}}(\vartheta)}\,\langle x, \mathcal{R}_{i+1}(\vartheta^{i+1}; Y_0)\,y\rangle \qquad (x \in H(Y_0),\ y \in T).$$

Status: read from the proof.

Problem 59.4.1. In a triangle of size m with (F), (D), (A), let $T' \subseteq H_\circ$ be a subspace, $\mathcal{E}_i$ endomorphisms of T' and C_i scalars with $g_0\langle \lambda_{i,0}x, \lambda_{i,0}y\rangle = C_i\langle x, \mathcal{E}_i y\rangle$ for $i < m$, $x \in H_\circ$, $y \in T'$. Then for $i + n = m$, $x \in H_\circ$ and $y \in T'$

$$\langle \lambda_i^{(n)}x, \lambda_i^{(n)}y\rangle = c_{(i)}\,\frac{g_0 g_1^n}{g_n}\,\langle x, \mathcal{E}_i\mathcal{E}_{i+1}\cdots\mathcal{E}_{m-1}\,y\rangle,$$

where $c_{(i)}$ is the product of $C_i, \ldots, C_{m-1}$ and of all $c_{i',n'}c^*_{i',n'}$ with $i' \geq i$.

Problem 59.4.2. For the triangle of the chain there is $c_\triangle \doteq \prod_{1 \leq i < j \leq m} \rho_{ij}$ with

$$\langle \lambda_0^{(m)}x, \lambda_0^{(m)}y\rangle = c_\triangle(\vartheta)\,\frac{g_0 g_1^m}{g_m\,c_{\mathrm{mat}}^m}\,\langle x, \mathcal{R}_\vartheta y\rangle \qquad (x \in H(Y_0),\ y \in T).$$

On $H(Y_0)$ we write Θ_ϑ for the lower chain with values in algebraic forms on Y_m, and on T for its restriction, as in Chapter 34. Its target consists of algebraic forms of the prescribed weight on Y_m that transform under a level, written $K_{\mathrm{last}}(\vartheta)$ in this sheet, by a character determined by ϑ. The group $K_{\mathrm{last}}(\vartheta)$ consists of the elements of the group $K^\circ(Y_m)$ of integral finite adelic unitary matrices of Y_m that reduce to 1 at each $w \in \mathcal{M}$ and to a block upper triangular matrix, with blocks $r, t, \dots, t$, at each $w \in Q_\vartheta$. The proof shows $[K^\circ(Y_m) : K_{\mathrm{last}}(\vartheta)] = \prod_{w\in\mathcal{M}} \#\mathrm{GL}_{r+mt}(k_w) \prod_{w\in Q_\vartheta} [\mathrm{GL}_{r+mt}(k_w) : \mathrm{Par}_w]$, with Par_w the group of Problem 59.3.6 over k_w, and that for $x, y \in H(Y_0)$

$$\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c_{\mathrm{wt}}\, [K^\circ(Y_m) : K_{\mathrm{last}}(\vartheta)]\, \langle \lambda_0^{(m)} x, \lambda_0^{(m)} y\rangle,$$

with c_{wt} a positive integer independent of ϑ. *Status*: read from the proof.

Problem 59.4.3. Θ_ϑ is injective on $H(Y_0)$, in particular on T.

Problem 59.4.4. For every m-fold twist ϑ,

$$[K^\circ(Y_m) : K_{\mathrm{last}}(\vartheta)]\, \frac{g_0 g_1^m}{g_m} = \big(c_{\mathrm{mat}}(\vartheta)\, c_{Q_\vartheta}\big)^m\, (\mathrm{Nm}\, Q_\vartheta)^{-mrt - t^2 m(m-1)/2}$$

times a positive number independent of ϑ.

Problem 59.4.5. There is c with $c \doteq \prod_{1\le i<j\le m} \rho_{ij}$ and $\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta)\, c_{Q_\vartheta}^m\, \langle x, \mathcal{R}_\vartheta y\rangle$ for all m-fold twists ϑ and $x, y \in T$.

Problems 59.2.3–59.2.7, 59.3.3 and 59.4.1–59.4.5, with the realization of the triangle, the first-edge identity and the description of the last level, prove the following.

Theorem 59.4.1. *There is a function c from m-fold twists to $\mathbb{C}$ with $c \doteq \prod_{1\le i<j\le m} \rho_{ij}$ such that for every m-fold twist ϑ and all $x, y \in T$,*

$$\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta)\, c_{Q_\vartheta}^m\, \langle x, \mathcal{R}_\vartheta y\rangle.$$

The data satisfy $t \le \Delta$ and $m\Delta \le 4b$ (as $m\Delta \le 4\sqrt{x}\,(3x^5)^8 < x_1 < b$ by Definition 33.3.1).

Problem 59.4.6. $\rho_{ij}(1) = 1$. The character μ_j^0/μ_i^0 has weight $-(j-i)\Delta$, with $t \le (j-i)\Delta \le 4b$, and tame part (ω_w^{j-i}), nontrivial exactly at the $w \in \mathcal{M}$; so each $P_z := (\mu_j^0/\mu_i^0, z)$, $1 \le z \le t$, is a ramified critical pair (Definition 34.10.1), and $\rho_{ij}(\vartheta) = \prod_{z=1}^t R_{P_z}(\vartheta^{j-i}, 1)$.

Problem 59.4.6 identifies the unprimed ratios of Problem 55.5.1; Theorem 55.5.2, used in Chapter 54, bounds for a family with small ratios the height of the inverses of $(\mathrm{Nm}\, Q_{\theta_\iota})^{e/2} \prod_{i<j} \rho_{ij}(\theta_\iota)$, every $e \in \mathbb{Z}$. We have proved Theorem 59.4.1 from three statements read from the proof; the exchange identity enters through the square laws. Chapter 55 uses it to factor $Z(\vartheta)/Z(1)$.

In the Lean formalization. In the Lean proof [GL] (`grh/src/New/Determinant/Chain/`), Theorem 59.4.1 is `New.Data.exists_inner_chain_eq`; ρ_{ij}, $\mathcal{R}_\vartheta$, $\doteq$ are `quotientRatio`, `rallisProduct`, `HasAbsUpToNormPower` (for the indices used here). The realization is `exists_isTriangle` (flag `false`), using `exists_triangleLift_triangleTransposedLift_comm` for (D) and `exists_triangleTransposedLift_adjoint_comm` for the identity in (A). The first-edge identity is a step in the proof of `exists_inner_weightedChain_eq`. The comparison at the last level is `inner_lowerChainAtLevel`, and the product formula for the index of $K_{\mathrm{last}}(\vartheta)$ is `stdLevelIndex_lastLevel_eq_prod`, both in `ChainNorms.lean`.

Hints

59.2.4: For injectivity, induction on n, passing a kernel vector to row $i+1$ by Problem 59.2.3 and (F). For $c_i^{[n]} \neq 0$, take a nonzero vector of $H_\circ$; a quotient by 0 is 0.

59.2.5: Induction on n; for the step use (A).

59.2.6: Problem 59.2.5 with $v = \lambda_i^{(n+1)} x$, then Problem 59.2.3 for $\lambda_{i+1}^{(n)} x'$.

59.3.2: The weights κ_j of Chapter 34 increase by Δ. For the rank bound use $r \geq 3$, $t = 5r$, $\Delta = 2r^8$; for nontriviality use $0 < n+1 \leq m < q_2$ and the m-fold condition.

59.3.3: Problem 58.5.5 with $S = \mathcal{M}$, $Q = Q_\vartheta$, $h = t$; the untwisted value $\mathrm{ex}_t(\mathcal{M}, \nu_{i,n}(1))$ is a constant. Note $\#\mathfrak{R} = \mathrm{Nm}\, Q_\vartheta \prod_{w \in \mathcal{M}} q_w$.

59.4.1: Induction on n from row $i+1$ to row i, with Problem 59.2.6, and (F) to apply the first-edge relation to x and $\mathcal{E}_{i+1} \cdots \mathcal{E}_{m-1} y$.

59.4.2: Problem 59.4.1 with $\mathcal{E}_i = \mathcal{R}_{i+1}(\vartheta^{i+1}; Y_0)/c_{\mathrm{mat}}$, and Problems 59.2.7 and 59.3.3.

59.4.4: Factor over $w \in \mathcal{M} \cup Q_\vartheta$; at $w \in Q_\vartheta$ use Problem 59.3.6, the stabilizer of an injective $(r+t) \times r$ matrix, and Problem 56.3.2.

59.4.5: Problems 59.4.2 and 59.4.4; for $A \in \mathbb{C}$ and $k \in \mathbb{Z}$, $\vartheta \mapsto A(\mathrm{Nm}\, Q_\vartheta)^k$ satisfies $\doteq 1$.

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