THE EXCHANGE IDENTITY

Prerequisites. Tate's unfolding of the global zeta integral and its Euler product [CF, Ch. XV, §§4.4–4.5]; Poisson summation on $\mathbb{Z}^2$; for Hecke L-series see also [Neu, Ch. VII, §8]. From Chapter 34: tame characters, their weight and tame part (Section 34.6), and the partial Hecke values $L^S(z,\nu)$ (Definition 34.10.1). From Chapter 56: Problem 56.2.1; its theta integrals, and Chapter 12 (Fock models), are background only.

In the determinant of a chain (Chapter 55) each pair of lifts contributes ratios of Hecke L-values. They come from an identity $I^\sharp = c\, I^\flat$ between a theta integral over a large unitary group and one over a small group, whose scalar has $|c|^2 = C\mathrm{ex}_h(S,\nu)$, with $\mathrm{ex}_h(S,\nu) = (\mathrm{Nm}\, S)^{h^2} \prod_{z=1}^{h} |L^S(z,\nu)|^2$ and $C > 0$ independent of the character. Why this factor? Removing a hyperbolic plane lowers the index of the exchange factor by one, at the cost of one completed L-value, and a separate equal-rank calculation supplies the equal-rank part of the normalization. The instances are of matching, of ex_h over $\mathbb{Q}(\mathrm{i})$ and of one step; there is none of the identity itself.

58.1. Matched functions

Problem 58.1.1. Over $\mathbb{F}_3$ let χ be the quadratic character, $\Psi(a,c) := \chi$ of the first nonzero entry of (a,c), $\Psi(0,0) := 0$, and $\Phi(x,y) := \Psi(y,-x)$. Then

$$\Phi(\text{last row of } k^{-1}) = \chi(\det k)\, \Psi(\text{first column of } k)$$

for all 48 matrices $k \in \mathrm{GL}_2(\mathbb{F}_3)$, and the common value is $+1$ for 24 of them.

Problem 58.1.2. Let A be a commutative ring, $n = n^\flat + n^\sharp$, and $k \in \mathrm{GL}_n(A)$. For a set I of $n^\flat$ rows, the minor of the first $n^\flat$ columns of k on the rows I is $\pm \det k$ times the minor of the last $n^\sharp$ rows of k^{-1} on the columns not in I. The last $n^\sharp$ rows of k^{-1} times the first $n^\flat$ columns of k is 0; over a field the row span of the former and the column span of the latter are each other's annihilators.

Definition 58.1.3. Let A be a finite commutative ring, χ a character of $A^\times$, and $\sigma\colon \mathcal{B} \sqcup \mathcal{A} \to \mathcal{N}$ a bijection of finite sets (in this sheet $\mathcal{N}$, $\mathcal{A}$, $\mathcal{B}$ are coordinate sets), $\#\mathcal{B} = n^\flat$, $\#\mathcal{A} = n^\sharp$; for $k \in \mathrm{GL}_{\mathcal{N}}(A)$ write $k_{\mathcal{B}}$ for its columns indexed by $\sigma(\mathcal{B})$ and $(k^{-1})_{\mathcal{A}}$ for the rows of k^{-1} indexed by $\sigma(\mathcal{A})$. Functions Φ on $\mathcal{A} \times \mathcal{N}$ matrices and Ψ on $\mathcal{N} \times \mathcal{B}$ matrices over A are *matched* for (σ,χ) if Φ vanishes off the matrices $(k^{-1})_{\mathcal{A}}$, Ψ vanishes off the matrices $k_{\mathcal{B}}$, and $\Phi((k^{-1})_{\mathcal{A}}) = \chi(\det k)\, \Psi(k_{\mathcal{B}})$ for every $k \in \mathrm{GL}_{\mathcal{N}}(A)$.

Problem 58.1.1 gives a matched pair with $n^\flat = n^\sharp = 1$.

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58.2. The two theta integrals

Let E be a CM field with a CM type and Weil data as in the proof (the field E of the data is one), υ_E (in this sheet) the splitting character of the Weil data on the ideles of E, and υ its restriction to the norm-one ideles, as in Section 34.7. For a character ν of the norm-one idele classes put $\tilde{\nu}(x) := \nu(x/\bar{x})$. As in Section 34.6, with μ in place of 0, ν has *weight* $\mu\colon \{\text{infinite places}\} \to \mathbb{Z}$ and *tame part* (ψ_w), characters of $k_w^\times$ almost all trivial, if $\nu(u) = \prod_\tau u_\tau^{\mu(\tau)} \prod_w \psi_w(u_w \bmod w)$ for every norm-one idele u with integral finite part.

Definition 58.2.1. A *modulus* is a nonzero ideal $\mathfrak{m} \subseteq \mathcal{O}_E$ such that no prime w divides $\mathfrak{m}$ together with its conjugate $\bar{w}$; $S_\mathfrak{m}$ is the set of primes dividing $\mathfrak{m}$. A character ν is *tame for* $(\mathfrak{m}, \mu, \chi)$, where χ is a character of $(\mathcal{O}_E/\mathfrak{m})^\times$, if $S_\mathfrak{m} \neq \varnothing$ and ν has weight μ and a tame part (ψ_w) with $\psi_w \neq 1$ exactly for $w \in S_\mathfrak{m}$ and $\chi(y) = \prod_{w \in S_\mathfrak{m}} \psi_w(y \bmod w)$.

Let V^0, $V^\sharp$, $V^\flat$ be framed Hermitian spaces over E (Section 34.3) on coordinate sets $\mathcal{N}$, $\mathcal{A}$, $\mathcal{B}$, of ranks n_0, $n^\sharp$, $n^\flat$, with a bijection $\sigma\colon \mathcal{B} \sqcup \mathcal{A} \to \mathcal{N}$ (so $n_0 = n^\sharp + n^\flat$). Assume: the number q_- of negative signs of V^0 is the sum of those of $V^\sharp$ and $V^\flat$; $n^\sharp = n^\flat + 2h$ with $h \in \mathbb{N}$; $2q_- + 2 \leq n_0$; and $n_0 \leq 2\mu(\tau) + 2q_-$ for every infinite place τ, for an integer-valued μ. For a Schwartz vector x of the archimedean Fock model of $(V^0, V^\sharp)$ and a function Φ on $\mathcal{A} \times \mathcal{N}$ matrices over $\mathcal{O}_E/\mathfrak{m}$, $[\Phi, x]$ is the adelic datum equal to x at the infinite places, to Φ read modulo $\mathfrak{m}$ on the matrices integral at $S_\mathfrak{m}$ (and 0 on the others) at the places of $S_\mathfrak{m}$, and to the vacuum elsewhere; likewise $[\Psi, v]$ for $(V^\flat, V^0)$. Put $\nu^0 := \nu^{-1}\upsilon^{n_0/2}$, a character for V^0; characters $\nu^\sharp$ for $V^\sharp$ and $\nu^\flat$ for $V^\flat$ are *compatible with* ν if $\tilde{\nu}^\sharp \upsilon_E^{-n^\sharp} \cdot \tilde{\nu}^\flat \upsilon_E^{-n^\flat} = \tilde{\nu}^{-1}$, equivalently $\nu^\sharp \nu^\flat = \nu^0$. The two functions of $g \in \mathrm{U}(V^0)(\mathbb{A})$ to be compared are

$$I^\sharp(g) := \int_{[\mathrm{U}(V^\sharp)]} \theta^{V^0, V^\sharp}_{\nu^0, \nu^\sharp}[\Phi, x](g, h_1)\, dh_1, \quad I^\flat(g) := \int_{[\mathrm{U}(V^\flat)]} \theta^{V^\flat, V^0}_{\nu^\flat, \nu^0}[\Psi, v](h_2, g)\, dh_2,$$

integrals of theta kernels over the quotients with their quotient measures. Theorem 58.5.1 gives, for each v, an x making the two integrals proportional on matched data for every compatible pair of characters.

Problem 58.2.2. The last two hypotheses say $\mu(\tau) \geq n_0/2 - q_- \geq 1$ for all τ. The ranks $(n_0, n^\sharp, n^\flat; q_-) = (2,1,1;0), (4,3,1;1), (6,5,1;2)$, with all q_- negative signs in $V^\sharp$, satisfy the hypotheses for every $\mu \geq 1$, with $h = 0, 1, 2$, and $n_0/2 - q_- = 1$ each time; these successive signatures can be realized by adjoining a hyperbolic plane to V^0 and to $V^\sharp$. For $n^\flat = 0$ the smallest case with $h \geq 1$ is $(2,2,0;0)$.

For a finite set S of finite places put $\mathrm{Nm}\, S := \prod_{w \in S} q_w$, $q_w = \#k_w$. Recall (Definition 34.10.1) that for $z > 1$, $L^S(z, \nu)$ is the product of $(1 - \tilde{\nu}_w(\varpi_w) q_w^{-z})^{-1}$ over $w \notin S \cup \bar{S}$, at chosen uniformizers ϖ_w, and at $z = 1$ its limit from the right whenever that limit exists and is finite. For finite S and ν with a tame part and a weight that is not identically zero, the values $L^S(z, \nu)$, $z \geq 1$ real, are finite and nonzero: for $z > 1$ the Euler product converges absolutely, and at $z = 1$ this is Hecke's nonvanishing on the line $\mathrm{Re}\, s = 1$ [La, Ch. XV]. *Status*: classical; the proof establishes it in this form.

Definition 58.2.3. For $h \in \mathbb{N}$, and for real $z \geq 1$, whenever the partial values in the formulas exist and are finite, the *exchange factor* and the *completed value* are

$$\mathrm{ex}_h(S,\nu) := (\mathrm{Nm}\, S)^{h^2} \prod_{z'=1}^{h} |L^S(z',\nu)|^2, \qquad \hat{L}_S(z,\nu) := (\mathrm{Nm}\, S)^{z-1/2} |L^S(z,\nu)|,$$

and $\mathrm{ex}_{-h}(S,\nu) := \mathrm{ex}_h(S,\nu)^{-1}$ when this is defined.

Problem 58.2.4. $\mathrm{ex}_0(S,\nu) = 1$. For $h \in \mathbb{N}$, if the values $L^S(j,\nu)$ are finite for the integers $1 \leq j \leq h+1$ and nonzero for $1 \leq j \leq h$, then $\mathrm{ex}_{h+1}(S,\nu)/\mathrm{ex}_h(S,\nu) = \hat{L}_S(h+1,\nu)^2$.

Problem 58.2.5. Assume $L^S(j,\nu)$ is finite and nonzero for every integer $j \geq 1$. For $d \in \mathbb{Z}$ let $z_d := d+1$ if $d \geq 0$ and $z_d := -d$ if $d \leq -1$. Then $z_d \geq 1$ and $\mathrm{ex}_{d+1}/\mathrm{ex}_d = \hat{L}_S(z_d,\nu)^2$ for every $d \in \mathbb{Z}$.

58.3. The factor over $\mathbb{Q}(\mathrm{i})$

In this section only, the field is $\mathbb{Q}(\mathrm{i})$ and α runs over $\mathbb{Z}[\mathrm{i}] \smallsetminus 0$. The Hecke character with $\tilde{\nu}((\alpha)) := (\alpha/\bar{\alpha})^4$ is unramified at all finite places and trivial on the ideles of $\mathbb{Q}$, so it has the form $x \mapsto \nu(x/\bar{x})$ for a character ν of the norm-one ideles, as the notation says; its L-function is $L(z) := \frac{1}{4}\sum_\alpha \alpha^8 |\alpha|^{-2z-8}$ for real $z > 1$.

Problem 58.3.1. For $y > 0$, $F_8(y) := \sum_\alpha \alpha^8 e^{-\pi y|\alpha|^2}$ satisfies $F_8(1/y) = y^9 F_8(y)$, and with $G_s(y_0) := \int_1^\infty e^{-y_0 y} y^s \, dy/y$ for $y_0 > 0$,

$$L(z) = \frac{\pi^{z+4}}{4\,\Gamma(z+4)} \sum_\alpha \alpha^8 \big[G_{z+4}(\pi|\alpha|^2) + G_{5-z}(\pi|\alpha|^2) \big]$$

for $z > 1$; with $1/\Gamma$ read as an entire function, the right side is defined for all real z. By machine, $L(1) \approx 1.5012288658$ and $L(2) \approx 1.2576665537$.

Let $\mathfrak{p} := (3+2\mathrm{i})$, so that $a + b\mathrm{i} \equiv a + 5b \pmod{\mathfrak{p}}$ and $a - b\mathrm{i} \equiv a - 5b$, and let ψ be the cubic character of $\mathbb{F}_{13}^\times$ with $\psi(2) = e^{2\pi\mathrm{i}/3}$. The twisted character is $\tilde{\nu}'((a+b\mathrm{i})) := \tilde{\nu}((a+b\mathrm{i}))\,\psi\big((a+5b)/(a-5b)\big)$, and 0 if $13 \mid (a+5b)(a-5b)$; take $S = \{\mathfrak{p}\}$, so $\mathrm{Nm}\, S = 13$ and $L^S(z) := L^S(z,\nu') = \frac{1}{4}\sum_\alpha \tilde{\nu}'((\alpha))|\alpha|^{-2z}$ for $z > 1$.

Problem 58.3.2. $\tilde{\nu}'$ is well defined on ideals prime to 13. By machine, $|L^S(1)| \approx 3.3363841832$ and $|L^S(2)| \approx 1.4580689579$; hence $\mathrm{ex}_1 \approx 144.708972$, $\mathrm{ex}_2 \approx 675898.752$, and $\mathrm{ex}_2/\mathrm{ex}_1 = 13^3 |L^S(2)|^2 \approx 4670.745$.

Problem 58.3.3. By machine, $|L^S(z)|/|L(z)| \approx 2.2224354055$ at $z = 1$ and ≈ 1.1593446241 at $z = 2$. The two primes above 5 contribute the factor $|1 - u_5 5^{-z}|^2/|1 - u_5\psi(2)5^{-z}|^2 \approx 2.0102465324$, resp. 1.1487493619, with $u_5 := \tilde{\nu}((2+\mathrm{i}))$.

58.4. Removing a plane

In the comparisons and equal-rank ratios of this section and the next, the indexing character ν (briefly, a *character*) has the fixed weight μ of Section 58.2 and a specified tame part whose nontrivial components form a nonempty finite set S disjoint from $\bar{S}$. Here $\mu \geq 1$ everywhere, and $S = S_{\mathfrak{m}}$ for a triple of Definition 58.2.1. The kernel characters $\nu^\sharp, \nu^\flat$ range over all pairs compatible with ν. The proof has three moves, which we describe in outline; the precise vectors, standard data and normalizations are those of [GL]. Let $\mathbb{H}$ be a hyperbolic plane, $V_+^0 := \mathbb{H} \oplus V^0$, and

$(V^\flat)^-$ the space $V^\flat$ with the opposite form, as in Chapter 56. *First*, at the top the common space is $\mathbb{H}^{n^\flat} \oplus V_+^0$ and the partners are $(V^\flat)^- \oplus V_+^0$ and $(V^\flat)^-$; from prescribed Schwartz vectors the proof builds vectors for the top pairs, and unfolding central Fourier coefficients and a vanishing criterion for singular forms give a fixed real λ and, for each character, $c \in \mathbb{C}$ with $T_{\mathrm{L}} = c\,T_{\mathrm{S}}$ on standard row and column data, where $T_{\mathrm{L}}, T_{\mathrm{S}}$ are intermediate large and small theta integrals (both integrated in the partner variable), and $\lambda|c|^2 = \mathrm{ex}_{n_0/2+1}(S,\nu)\,\mathrm{er}$, er the equal-rank ratio of Section 58.5. This move is not developed here. *Second*, the proof removes $n^\flat + 1$ planes, keeping $(V^\flat)^-$ fixed. At each step it takes both constant terms along an isotropic line of the common space and uses a large partner with the prescribed signs and an adapted splitting. The index decreases by one at each step, from $n_0/2 + 1$ to h (Problems 58.4.1–58.4.4). The proof then passes to matched functions and to the adjoint model, lifts the data modulo $\mathfrak{m}$, and twists both kernels, to obtain $I^\sharp$ and $I^\flat$ for every compatible pair. *Third*, a vector with nonzero constant er is found (Section 58.5). (The letters λ, λ', λ_i, c_{pl}, C_{pl}, c_{sl}, the index d, and the spaces W, V, Y, J of this section and the next are local to this sheet.)

Problem 58.4.1. In $\mathbb{F}_9^2$ with the Hermitian form $N(u) - N(v)$, N the norm to $\mathbb{F}_3$, the average over $c \in \mathbb{F}_3$ of $e^{2\pi i c(N(u)-N(v))/3}$ is 1 at the 33 isotropic vectors of Problem 56.2.1 and 0 elsewhere. Of the 9^4 matrices $w \in M_2(\mathbb{F}_9)$, 81 satisfy $(1,1)w = 0$, and none of these is invertible; $\#\mathrm{GL}_2(\mathbb{F}_9) = 5760$.

Problem 58.4.2. Let $\mathbb{F}$ be a finite field, $\mathbf{e}$ a nontrivial additive character of $\mathbb{F}$, $y \in \mathbb{F}^{n^\flat}$ nonzero, and ℓ a linear form on functions on $M_{n^\flat \times n_0}(\mathbb{F})$ such that $\ell(f) = \ell(f_a)$, $f_a(w) := f(w)\,\mathbf{e}(y^{\mathsf{T}} w a)$, for every f and every $a \in \mathbb{F}^{n_0}$. Then $\ell(f) = 0$ for every f supported on the w with $y^{\mathsf{T}} w \neq 0$, in particular for every f supported on matrices of rank $n^\flat$.

In the proof the small constant term is a sum over rational vectors of $(V^\flat)^-$; the central average (as in Problem 58.4.1) and the vanishing of the zero term on full-rank data (as in Problem 58.4.2) leave only the index zero [GL].

Problem 58.4.3. Let E be any CM field, ν a unitary character of its norm-one idele classes, S a finite set of finite places, $z > 1$ real, and $f = \otimes_w f_w$ Schwartz–Bruhat on $\mathbb{A}_E$; the quotient measure is induced by $d^\times y$ and counting measure. Then

$$\int_{\mathbb{A}_E^\times / E^\times} \sum_{a \in E^\times} f(ay)\,\tilde{\nu}(y)|y|^z\,d^\times y = \int_{\mathbb{A}_E^\times} f(y)\,\tilde{\nu}(y)|y|^z\,d^\times y,$$

and if $\tilde{\nu}$ is unramified and $f_w = 1_{\mathcal{O}_w}$ at every finite $w \notin S \cup \bar{S}$, the finite part of the right side is $L^S(z,\nu)$ times the integrals of $f_w \tilde{\nu}_w |\cdot|^z$ at $w \in S \cup \bar{S}$ (Haar measures with $\mathrm{vol}(\mathcal{O}_w^\times) = 1$). In Problem 58.3.1, $E = \mathbb{Q}(\mathrm{i})$, $S = \varnothing$, $f_\infty(t) = t^8 e^{-\pi|t|^2}$, $f_w = 1_{\mathcal{O}_w}$, and the finite factor is $L(z)$.

On the large side the zero vector is killed by the datum and the surviving line orbit unfolds as in Problem 58.4.3. In outline, the *one-plane formula* reads as follows (the standard row and column data, built from an integral matrix at S and $\tilde{\nu}^{-1}$ of a determinant, the adapted frames and the cone vectors are specified in [GL]). Let the common space be $\mathbb{H} \oplus W$ and the large partner $\mathbb{H} \oplus V$, with adapted splittings, the small partner $(V^\flat)^-$ fixed, $\mathrm{rk}\,W = \mathrm{rk}\,V + n^\flat$ even, and $d := \mathrm{rk}\,V - \mathrm{rk}\,W/2$. There is $C_{\mathrm{pl}} > 0$ independent of the character and, for each character, $c_{\mathrm{pl}} \in \mathbb{C}$

with $|c_{\mathrm{pl}}| = C_{\mathrm{pl}}\hat{L}_S(z_d,\nu)$, z_d as in Problem 58.2.5, such that on the standard row data, for every Schwartz vector x and $g \in \mathrm{U}(W)(\mathbb{A})$, the constant term of the large integral at the Levi point $(1,g)$ is c_{pl} times the downstairs large integral at g of the cone vector of x (the dual cone if $d \le -1$, where Poisson summation precedes evaluation, so that only values of L^S at points ≥ 1 occur). On the standard column data the small constant term at $(1,g)$ is c_{sl} times the downstairs small integral of the zero slice, $c_{\mathrm{sl}} \ne 0$ a fixed vacuum factor. *Status*: for cuspidal input Rallis proves the tower property for the orthogonal–symplectic pair [Ra84, Thm. I.1.1], and Wu gives the unitary tower property [Wu, Prop. 3.1]; the proof computes the contribution of the surviving line orbit for the character and the ramified standard data [GL].

Problem 58.4.4. On the standard data of the one-plane formula, suppose $T_{\mathrm{L}} = c\,T_{\mathrm{S}}$ upstairs with $\lambda|c|^2 = \mathrm{ex}_{d+1}\,\mathrm{er}$. Then downstairs, for the cone vector and the zero slice, $T_{\mathrm{L}} = c'\,T_{\mathrm{S}}$ with $c' = c\,c_{\mathrm{sl}}/c_{\mathrm{pl}}$ and $\lambda'|c'|^2 = \mathrm{ex}_d\,\mathrm{er}$, where $\lambda' = \lambda C_{\mathrm{pl}}^2/|c_{\mathrm{sl}}|^2$ does not depend on the character (note $c_{\mathrm{pl}} \ne 0$).

58.5. The constant of equal rank

After the planes are removed, $\lambda|c|^2 = \mathrm{ex}_h(S,\nu)\,\mathrm{er}$, where the *equal-rank ratio* of a Schwartz vector v_+ for (V_+^0, V_+^0) is $\mathrm{er} = |I_+(v_+)(1)|^2/\mathrm{ex}_{n_0/2+1}$, with $I_+(v_+)$ the large theta integral of (V_+^0, V_+^0), with $\nu^{-1}\upsilon^{n_0/2+1}$ in both slots, for v_+ and the standard function equal to $\tilde{\nu}^{-1}(\det)$ on $\mathrm{GL}_{n_0+2}(\prod_{w\in S}\mathcal{O}_w)$ and 0 elsewhere. This integral is constant in the group variable, so er is a number. More generally, for an even-rank space Y the ratio for (Y,Y) uses the same construction with $\mathrm{rk}\,Y$ in place of n_0+2; for $Y = \mathbb{H}\oplus J$ with $\mathrm{rk}\,J = 2k_0$ its denominator is $\mathrm{ex}_{k_0+1}(S,\nu)$. For the rest of this section fix $k_0 \in \mathbb{N}$ with $k_0 \le \mu(\tau)$ at every infinite place. The next two facts enter the proof that a suitable v_+ has a nonzero constant ratio. *Continuation*: for a character with its set S, the values $L^S(z,\nu)$, $z \ge 1$, extend real-analytically to $(-k_0,\infty)$, nonzero at every integer $j > -k_0$; for integers $-k_0 < j \le 0$ the functional equation in absolute values relates j to $1-j \ge 1$, where the nonvanishing of Section 58.2 applies. For the classical continuation and functional equation see [CF, Ch. XV, §§4.4–4.5]. *Descent of vanishing*: let J be the standard positive definite space of rank $2k_0$; if at one character the equal-rank ratio on $\mathbb{H}\oplus J$ vanishes for every Schwartz vector, then the large integral for $(\mathbb{H}^{2k_0}\oplus\mathbb{H}\oplus J, J^-\oplus\mathbb{H}\oplus J)$ vanishes on every Schwartz vector and standard row datum, and by the continuation this descends through $2k_0+1$ planes to $(J,0)$; this is read from the proof and not developed.

Problem 58.5.1. For the twisted character of Problem 58.3.2 (ramified at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, so L^S is its full L-function), by machine the continuation has $|L^S(0)| \approx 55.2242116195$ and $|L^S(-1)| \approx 6198.83038564$.

Fix a Schwartz vector for $(\mathbb{H}\oplus J, \mathbb{H}\oplus J)$ and let er be its equal-rank ratio. Exchanging once with an empty small partner, $(V^0, V^\sharp, V^\flat) = (J,J,0)$, and once with an empty large partner, $(J,0,J)$ (a comparison of the proof with index $-k_0$, outside the range $h \ge 0$ of Section 58.2), the proof finds a Schwartz vector y for (J,J) and numbers $\lambda_1,\lambda_2 \in \mathbb{R}$, $\lambda_0 \in \mathbb{C}$ independent of the character, and for each character numbers c_1, c_2 with $\lambda_1|c_1|^2 = \mathrm{ex}_{k_0}\mathrm{er}$, $\lambda_2|c_2|^2 = \mathrm{ex}_{-k_0}\mathrm{er}$, and $c_1c_2 = \lambda_0$.

Problem 58.5.2. Under these relations $\mathrm{er}^2 = \lambda_1\lambda_2|\lambda_0|^2$, independently of the character.

Problem 58.5.3. For the pair $(J,0)$, the vacuum 1 in the one-dimensional Fock space of $\mathrm{Hom}(J,0)$ and the standard datum equal to 1 at the empty matrix, the theta integral over $[\mathrm{U}(0)]$ at $g=1$ is $\mathrm{vol}[\mathrm{U}(0)]\neq 0$. Given one character, some Schwartz vector on $\mathbb{H}\oplus J$ has nonzero equal-rank ratio at it; and some Schwartz vector has equal-rank ratio equal to one positive constant for every character (also when there is no character).

Now take $k_0 = n_0/2 - q_-$, allowed by Problem 58.2.2. Each time a plane is added the proof finds a new Schwartz vector whose ratio is a positive multiple, independent of the character, of the preceding one; after q_- steps and a transfer to the space with the signs of V^0_+ there is v_+ with $\mathrm{er} = C_0$ for every character, $C_0\neq 0$.

Problem 58.5.4. Fix this C_0 and the real coefficient λ of the comparison, both independent of $(\mathfrak{m},\nu,\chi)$, and suppose $\lambda|c|^2 = \mathrm{ex}_h(S,\nu)C_0$ for every character. If a character exists then $c\neq 0$, $\lambda\neq 0$, and $C := C_0/\lambda = |c|^2/\mathrm{ex}_h > 0$ does not depend on $(\mathfrak{m},\nu,\chi)$.

Theorem 58.5.1. *Under the hypotheses of Section 58.2, for every Schwartz vector v of the real Fock model of $(V^\flat,V^0)$ there are a real $C>0$ and a Schwartz vector x of the real Fock model of $(V^0,V^\sharp)$ such that for every modulus $\mathfrak{m}$, every character ν of the norm-one idele classes and every character χ of $(\mathcal{O}_E/\mathfrak{m})^\times$ for which ν is tame for $(\mathfrak{m},\mu,\chi)$, there is $c\in\mathbb{C}$ with*

$$|c|^2 = C\,\mathrm{ex}_h(S_{\mathfrak{m}},\nu)\neq 0$$

and $I^\sharp(g) = c\,I^\flat(g)$ for all characters $\nu^\sharp,\nu^\flat$ compatible with ν, all (Φ,Ψ) matched for (σ,χ) over $\mathcal{O}_E/\mathfrak{m}$, and all $g\in\mathrm{U}(V^0)(\mathbb{A})$.

Here x and C are chosen before $\mathfrak{m},\nu,\chi$; only c depends on them. The theorem is taken from the proof [GL]. Problems 58.2.4, 58.2.5, 58.4.4, 58.5.2, 58.5.3 and 58.5.4 give the scalar calculations, using the comparisons, the continuation, the descent of vanishing and the constructions of equal-rank vectors stated above. The proof adapts a criterion of Freitag for singular forms [Fr]; for singular forms and theta lifts see also [Li97].

Problem 58.5.5. Let S and Q be disjoint finite sets of finite places, $h\in\mathbb{N}$, and ν,ν' characters whose values $L^S(z,\nu)$ and $L^{S\cup Q}(z,\nu')$ are finite, with $L^S(z,\nu)\neq 0$, for each integer $1\le z\le h$. Then

$$\mathrm{ex}_h(S\cup Q,\nu') = \mathrm{ex}_h(S,\nu)\,(\mathrm{Nm}\,Q)^{h^2}\prod_{z=1}^{h}\frac{|L^{S\cup Q}(z,\nu')|^2}{|L^S(z,\nu)|^2}.$$

So $|c|^2 = C\prod_{j=1}^{h}\hat{L}_S(j,\nu)^2$: the index drops by one with each plane removed, and, when a character exists, the equal-rank value and the comparison coefficient give $C = C_0/\lambda$. Chapter 59 applies Theorem 58.5.1 to the squares of a triangle of lifts, with $S = \mathcal{M}\cup Q_\vartheta$, and uses Problem 58.5.5 to divide by the untwisted value.
In the Lean formalization. In the Lean proof [GL] (readable edition; names relative to `New`): Theorem 58.5.1 is `Exchange.exchange_identity` in `New/Determinant/Exchange/TheoremForPartners.lean`; Definitions 58.2.1 and 58.1.3 are `Modulus`, `Exchange.IsTameFor` and `AreMatched`; ex_h, ex_{-h} and $\hat{L}_S$ are `Exchange.lValueProd`, `Exchange.lValueProdInt` and `Exchange.lSharp`. In `New.Exchange`: the top comparison is `exists_rowColExchangeIdentity_sum`; the large side of the one-plane formula is

`exists_planeConstantTerm_eq_thetaLarge_cone` with its `_dualCone` variant, and plane removal is `exists_rowColExchangeIdentity_removing_plane`; the continuation is `exists_analytic_extension_ne_zero`; the equal-rank constant is `exists_equalRankThetaRatio_eq_const`. The natural-index factor `lValueProd` takes S and ν separately; `lValueProdInt` and `lSharp` take a split tame character with its ramified set.

Hints

§58.1.

58.1.1: The last row of k^{-1} is $(\det k)^{-1}(-c, a)$; χ is quadratic.

58.1.2: For $I = \{1, \dots, n^\flat\}$, replace the first $n^\flat$ rows of k^{-1} by those of the identity and multiply by k.

§58.2.

58.2.2: Use $n_0 = n^\sharp + n^\flat$ and subtract $2q_-$ from the rank bounds.

58.2.5: For $d = -j$, apply Problem 58.2.4 with $h = j - 1$.

§58.3.

58.3.1: Poisson summation ($(a + b\mathrm{i})^8$ is harmonic); split the Mellin integral at $y = 1$. By machine: mpmath's `gammainc`.

58.3.2: $\psi(-1) = 1$. By machine: as in Problem 58.3.1, with the finite Fourier transform of the character modulo 13 on the dual side.

58.3.3: $\tilde{\nu}'((2 \pm \mathrm{i}))$ are $u_5\psi(2)$ and its conjugate.

§58.4.

58.4.1: Orthogonality of the characters of $\mathbb{F}_3$; for the count, $(1, 1)w = 0$ says that the rows of w are opposite.

58.4.2: Use orthogonality in the variable a.

58.4.3: Unfold; the finite integral is a product over places.

58.4.4: Take constant terms of both sides; then Problem 58.2.5.

§58.5.

58.5.1: The split Mellin formula with the finite Fourier transform of the twist, and the entire function $1/\Gamma$.

58.5.2: Compare the factors with indices k_0 and $-k_0$.

58.5.3: $\mathrm{Hom}(J, 0) = 0$. Argue by contradiction with the descent of vanishing; then Problem 58.5.2, $\mathrm{er} \geq 0$.

58.5.4: $\mathrm{ex}_h > 0$.

58.5.5: Split the product over $S \cup Q$.

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