EQUAL MULTIPLICITIES

Prerequisites. Linear algebra over $\mathbb{C}$: traces, characteristic polynomials and Newton's identities [Mac, Ch. I, §2]. Modular forms of weight two on $\Gamma_0(N)$ and their Hecke operators [DS, Ch. 5]. From Chapter 24, *Hecke operators and theta lifts*: Section 24.4, with Problems 24.4.2 and 24.4.3. From Chapter 34, *Spaces, forms and twists*: the chosen places $\mathcal{M}$, the group $\bar{G}$, good families of labels and the idempotent e_0 (Section 34.1); the gap Δ and the target parameter ς_τ (Section 34.2, Problem 34.2.1); the spaces Y_0, W_j and their signs (Sections 34.3 and 34.4, Problem 34.3.1); principal and good forms $H(V) \supseteq H_g(V)$ and the target $T = H_g(Y_0)$ of dimension d_T; tame characters (Section 34.6). From Chapter 33: $r = 3^{e'}$ is odd, $x^5 \le r$, m is even, and the weights $\kappa(\tau)$ are integers. From Chapter 56: the Rallis operators $\mathcal{R}_j(\xi; V)$ (Definition 56.3.2). Inner products are conjugate-linear in the first variable. Problems marked "by machine" use Sage [Sage].

The single lifts of Chapter 55, *The determinant of a chain*, start from the spaces W_j, which have the rank of Y_0 but other signs at the real places. How can a determinant on $H_g(W_j)$ be compared with one on $T = H_g(Y_0)$, when no map between the two spaces is given? In the proof one Hecke algebra acts on both spaces with equal traces. The equality gives their dimensions and characteristic polynomials and, with the decompositions into joint eigenspaces, the multiplicities of the Hecke eigensystems. This sheet derives the consequences, and states the equality (Theorem 57.3.3) with the comparison of trace formulas behind it.

57.1. Hecke operators of a finite group

Here K is a subgroup (the field $\mathbb{Q}(\sqrt{-d})$ does not occur). Let the symmetric group S_3 act on $\mathbb{C}^3$ by $(\sigma v)_i = v_{\sigma^{-1}(i)}$, let $K = \{1, (12)\}$ and $g = (23)$.

Problem 57.1.1. The double coset KgK has 4 elements. For a K-fixed vector $v = (v_1, v_1, v_3)$, the average of σv over $\sigma \in KgK$ is $\left(\frac{v_1+v_3}{2}, \frac{v_1+v_3}{2}, v_1\right)$; in the coordinates (v_1, v_3) of the K-fixed vectors it is $(v_1, v_3) \mapsto \left(\frac{v_1+v_3}{2}, v_1\right)$.

Problem 57.1.2. Let a group G act on a complex inner product space U by unitary operators, let $K \subseteq G$ be a subgroup and $g \in G$ be such that $K_g := K \cap gKg^{-1}$ has finite index in K. On the K-fixed vectors U^K the operator $\mathrm{av}_K(g) := [K : K_g]^{-1} \sum_{\delta K_g \in K/K_g} \delta g$ does not depend on the representatives, maps U^K to itself, and satisfies $\langle z, gy\rangle = \langle z, \mathrm{av}_K(g) y\rangle$ for all $z, y \in U^K$. If K is finite, $\mathrm{av}_K(g)y$ is the average of σy over $\sigma \in KgK$.

Problem 57.1.3. The operator of Problem 57.1.1 has traces $2, \frac{1}{2}, \frac{5}{4}, \frac{7}{8}$ in its powers $0, 1, 2, 3$, characteristic polynomial $(X-1)(X+\frac{1}{2})$ and determinant $-\frac{1}{2}$.

Problem 57.1.4. Let a monoid act linearly, with its unit acting as the identity and products as composites, on finite-dimensional complex spaces U and U', and suppose $\mathrm{tr}_U h = \mathrm{tr}_{U'} h$ for every element h. Then $\dim U = \dim U'$, and every

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element h has the same characteristic polynomial, hence the same determinant, on U and on U'.

Problem 57.1.5. The matrices $\mathrm{diag}(1,2)$ and $\left(\begin{smallmatrix} 0 & -2 \\ 1 & 3 \end{smallmatrix}\right)$ both have traces $3, 5, 9$ in their powers $1, 2, 3$, and $\det(h+1) = 6$ for both.

Problem 57.1.6. In Problem 57.1.4, suppose moreover that the monoid is commutative and acts diagonalizably on U and on U'. Then for every function λ from the monoid to $\mathbb{C}$ the joint eigenspaces $\{y : hy = \lambda(h)y \text{ for all } h\}$ in U and in U' have the same dimension.

57.2. Brandt matrices: a model

Where do two spaces with the same Hecke traces come from? The classical case compares a definite quaternion algebra with GL_2. Let B, $\mathcal{O}_1$, $\mathcal{O}_2$, the Brandt matrices $B(\ell)$ for primes $\ell \neq 11$, and the cusp form f be as in Section 24.4 and Problem 24.4.3; $B(\ell)$ acts on functions y on the two classes by $(B(\ell)y)(i) = \sum_j B_{ij}(\ell)\, y(j)$. The space of cusp forms of weight 2 on $\Gamma_0(11)$ is $\mathbb{C}f$ [DS, §3.9], and $T_\ell f = a_\ell(f) f$.

Problem 57.2.1. (By machine.) For $\ell = 2, 3, 5, 7, 13$: $\mathrm{tr}\, B(\ell) = 1, 3, 7, 6, 18$, and $\mathrm{tr}\, B(\ell) - (\ell+1) = a_\ell(f) = -2, -1, 1, -2, 4$.

Problem 57.2.2. (a) If an idempotent e on a finite-dimensional space U commutes with an operator h, then eU is stable under h and $\mathrm{tr}(h \mid eU) = \mathrm{tr}_U(he)$. (b) For a prime $\ell \neq 11$, the constant function is an eigenvector of $B(\ell)$ with eigenvalue $\ell+1$, and the line $U_0 := \{y : y(1)/|\mathcal{O}_1^\times| + y(2)/|\mathcal{O}_2^\times| = 0\}$ is stable under $B(\ell)$; hence $\mathrm{tr}(B(\ell) \mid U_0) = \mathrm{tr}\, B(\ell) - (\ell+1)$.

For the five primes of Problem 57.2.1, Problems 57.2.1 and 57.2.2 give $\mathrm{tr}(B(\ell) \mid U_0) = a_\ell(f)$: once the constants are removed, the Brandt matrices have the traces of the Hecke operators on cusp forms. The equality for every prime $\ell \neq 11$ is the level-11 case of the correspondence of Eichler and of Jacquet and Langlands [Ei73], [JL, §16]; *Status*: classical. It is a model only: the group of units of B and GL_2 differ at 11 as well as at infinity, whereas the proof compares unitary groups that differ only at the real places.

Problem 57.2.3. (By machine.) The quaternion algebra over $\mathbb{Q}$ ramified at 37 and ∞ has three classes of left ideals of a maximal order; its Brandt matrix $B(2)$ has characteristic polynomial $(X-3)X(X+2)$, and T_2 on the cusp forms of weight 2 on $\Gamma_0(37)$ has characteristic polynomial $X(X+2)$.

At level 37 the factor of the constants is $X-3$, and what remains is the characteristic polynomial of T_2 on cusp forms.

57.3. Equal traces

Let V be a framed space of rank r over E with numbered signs $s = (s(0), \ldots, s(r-1))$ (in this sheet s is a system of signs and, below, a an element of a group algebra; the numbers s and a of the data do not occur), such as Y_0 or W_j (Section 34.4). Through the frames, all such V have the same finite adelic unitary group G_f, that of the identity form; it acts on forms on V by right translation. Give $H(V)$ and its subspaces the *Petersson inner product* $\langle y', y\rangle := \int \overline{y'(g)}\, y(g)\, dg$, the integral over

the quotient of the adelic unitary group of V by its rational points, for the quotient measure fixed in the proof.

Definition 57.3.1. The numbered signs s of V satisfy the *gap hypothesis* if at every infinite place $\tau \in \Phi$ the entries $2\varsigma_\tau(0), \dots, 2\varsigma_\tau(r-1)$ are distinct, are congruent to $r-1$ modulo 2, and satisfy $|2\varsigma_\tau(i) - 2\varsigma_\tau(i')| \geq 2\Delta$ whenever $s(i) \neq s(i')$.

Under the gap hypothesis the proof shows that $H(V)$ is finite-dimensional; traces below are taken under this hypothesis.

Problem 57.3.1. The gap hypothesis holds for Y_0 and for W_j, $1 \leq j \leq m$.

Definition 57.3.2. In this sheet $G_f^{\mathcal{M}}$ is the subgroup of the $g \in G_f$ whose component at every $w \in \mathcal{M}$ is 1; the component at w determines that at $\bar{w}$, so these components are 1 as well. An element a of the group algebra $\mathbb{C}[G_f^{\mathcal{M}}]$ is *spherical* if, in every representation of G_f, it maps the vectors fixed by the principal level to vectors fixed by the principal level. A spherical a acts on $H(V)$ for every framed V of rank r; the *Hecke algebra away from* $\mathcal{M}$, $\mathbb{T}^{\mathcal{M}}$, is the algebra of the families of operators on the spaces $H(V)$, one for each V, given in this way by one spherical element. For a finite place w and $i \in \mathbb{N}$, let $h_{w,i} \in \mathbb{T}^{\mathcal{M}}$ be the Hecke operator (the sum of the translates by representatives of its single cosets) of the $\mathrm{GL}_r(\mathcal{O}_w)$-double coset of $\mathrm{diag}(\varpi_w 1_i, 1_{r-i})$ in $\mathrm{GL}_r(E_w)$, with $\mathcal{O}_w$ the local ring of integers and ϖ_w a uniformizer, if w is split, $w, \bar{w} \notin \mathcal{M}$ and $i \leq r$; in all other cases $h_{w,i} := 0$.

In this sheet an *eigensystem* is a family $\lambda = (\lambda_{w,i})$ of complex numbers indexed by the finite places w and $i \in \mathbb{N}$, and $H(V)_\lambda := \{y \in H(V) : h_{w,i} y = \lambda_{w,i} y \text{ for all } w, i\}$. The proof lets $\bar{G}$ act on $H(V)$ through integral lifts whose components outside $\mathcal{M} \cup \overline{\mathcal{M}}$ are 1.

Problem 57.3.2. An element of $G_f^{\mathcal{M}}$ commutes with every element of G_f whose components outside $\mathcal{M} \cup \overline{\mathcal{M}}$ are 1. Hence every $h \in \mathbb{T}^{\mathcal{M}}$ commutes with the action of $\bar{G}$, and so with e_0, on $H(V)$, and maps $H_g(V)$ to itself.

For V satisfying the gap hypothesis and $h \in \mathbb{T}^{\mathcal{M}}$, write $\mathrm{tr}(h \mid H_g(V))$ for the trace of the restriction of h to $H_g(V)$. The proof establishes the following.

Theorem 57.3.3. *Let V and V' be framed spaces of rank r over E whose numbered signs satisfy the gap hypothesis. Then every $h \in \mathbb{T}^{\mathcal{M}}$ has* $\mathrm{tr}(h \mid H_g(V)) = \mathrm{tr}(h \mid H_g(V'))$.

Theorem 57.3.3 is not proved here. The proof deduces it from a comparison of simple trace formulas. Let E be a CM field with a fixed CM type, containing the 5040th roots of unity and having an even number of infinite places. Let V, V' be framed spaces of rank $n \geq 2$ over E with numbered signs s, s', and G_f their common finite adelic group. For each infinite place τ let $l_\tau \in \mathbb{Z}^n$ have distinct entries congruent to $n-1$ modulo 2, with $|l_\tau(i) - l_\tau(i')| \geq 2(500n^3 + 10n^5)$ whenever $s(i) \neq s(i')$ or $s'(i) \neq s'(i')$. Let K_f be a subgroup of the group of integral elements of G_f that contains a neighbourhood of the identity, and let $a \in \mathbb{C}[G_f]$ map K_f-fixed vectors to K_f-fixed vectors in every representation of G_f. Suppose that the test function $a \star 1_{K_f}$ is *cuspidal* at a split place w, with residue field k_w: for every nonempty proper set I of coordinates, with pr_I the diagonal projection onto it, every $\bar{g} \in \mathrm{GL}_n(k_w)$, every family $(u_A)_{A \in M_n(k_w)}$ in $\mathrm{GL}_n(\mathcal{O}_w)$ with u_A reducing to $(1 + \mathrm{pr}_I A(1 - \mathrm{pr}_I))\bar{g}$, and every $g \in G_f$, the sum of $(a \star 1_{K_f})(g\, u_A)$ over $A \in$

$M_n(k_w)$ is zero, u_A being placed in G_f at w. Then operators acting as a on the K_f-fixed forms of parameter l on V and on V' (defined like the principal forms, with l_τ for $2\varsigma_\tau$ and K_f for the principal level) have the same trace. For the classical arguments, see [Lab, Thms. 5.4–5.5], [AC, Ch. 1, Lemmas 2.4–2.6] and, for a cuspidal component, [HL, Prop. 1.10.5]; *Status*: the method is classical; the proof establishes the comparison in this form, with integrable idempotent kernels of discrete series in place of pseudo-coefficients.

The data meet these conditions: the proof shows that E contains the 5040th roots of unity and has an even number of infinite places, and $\Delta = 2r^8 \geq 500r^3 + 10r^5$ and $r \geq 2$ because $r \geq x^5 \geq 9$, x being large. The idempotent e_0 is the sum of the orthogonal idempotents $e_c := \bigotimes_w e_{c_w}$ over the good families c, so by Problem 57.2.2(a) the trace of $h \in \mathbb{T}^{\mathcal{M}}$ on $H_g(V)$ is $\sum_c \mathrm{tr}_{H(V)}(he_c)$. The proof shows that he_c acts on $H(V)$ for every V through one element of $\mathbb{C}[G_f]$ that keeps the principal-level vectors, and that its test function is cuspidal at a chosen place w where c_w is primitive (irreducible with a root of order $(\#k_w)^r - 1$); every good family has such a place (Section 34.1), and chosen places are split. The comparison applies to each term, with K_f the principal level and $l_\tau = 2\varsigma_\tau$.

57.4. Dimensions, eigenspaces and determinants

Problem 57.4.1. $\dim H_g(W_j) = d_T$ for $1 \leq j \leq m$.

When $H(V)$ is finite-dimensional, the proof shows that it is the direct sum of the spaces $H(V)_\lambda$ (the operators $h_{w,i}$ are normal and commute).

Problem 57.4.2. Let λ be an eigensystem and fix finitely many eigensystems $\lambda' \neq \lambda$. A polynomial in finitely many of the $h_{w,i}$, an element of $\mathbb{T}^{\mathcal{M}}$, acts on $H(V)$, for every framed V of rank r, as the identity on $H(V)_\lambda$ and as zero on each of these $H(V)_{\lambda'}$.

Problem 57.4.3. For every eigensystem λ and $1 \leq j \leq m$, the spaces $T \cap H(Y_0)_\lambda$ and $H_g(W_j) \cap H(W_j)_\lambda$ have the same dimension.

Let ξ be a tame character and $j \in \mathbb{N}$. By Definition 56.3.2 and the formula for its matrix coefficients proved with it, for every framed V of rank r and all $z, y \in H(V)$,

$$\langle z, \mathcal{R}_j(\xi; V)\, y\rangle = \sum_\gamma \mathrm{mc}(\gamma)\, \langle z, \gamma y\rangle,$$

a sum, convergent in any order, over the countable family of representatives $\gamma \in G_f$ of that definition, the same family for every V, whose components at the places of $\mathcal{M}$ are 1, with the coefficients $\mathrm{mc}(\gamma)$ of Chapter 56, which depend on ξ, j and γ but not on V; here $\gamma y := y(\cdot\,\gamma)$. For each $\gamma \in G_f^{\mathcal{M}}$ the proof supplies an element $\mathrm{av}(\gamma) \in \mathbb{T}^{\mathcal{M}}$, an average of translates as in Problem 57.1.2, such that $\langle z, \mathrm{av}(\gamma) y\rangle = \langle z, \gamma y\rangle$ for every framed V of rank r and all $z, y \in H(V)$.

Problem 57.4.4. Let U and U' be finite-dimensional complex inner product spaces, and let pairs $(h_i, h_i') \in \mathrm{End}\, U \times \mathrm{End}\, U'$, for i in a countable set, lie in one complex linear subspace. Let $h_\infty \in \mathrm{End}\, U$ and $h_\infty' \in \mathrm{End}\, U'$ be such that $\sum_i \langle z, h_i y\rangle = \langle z, h_\infty y\rangle$ for all $z, y \in U$ and $\sum_i \langle z', h_i' y'\rangle = \langle z', h_\infty' y'\rangle$ for all $z', y' \in U'$, the sums converging in any order. Then (h_∞, h_∞') lies in that subspace. In particular, let a complex algebra act on U and on U' through complex-linear algebra homomorphisms to $\mathrm{End}\, U$ and $\mathrm{End}\, U'$; if each (h_i, h_i') is the action of one element, then one element acts as h_∞ on U and as h_∞' on U'.

Problem 57.4.5. Let V and V' be framed spaces of rank r satisfying the gap hypothesis, ξ a tame character and $j \in \mathbb{N}$. One element of $\mathbb{T}^{\mathcal{M}}$ acts as $\mathcal{R}_j(\xi; V)$ on $H(V)$ and as $\mathcal{R}_j(\xi; V')$ on $H(V')$. In particular $\mathcal{R}_j(\xi; V)$ maps $H_g(V)$ to itself.

Problem 57.4.6. For $1 \le j \le m$ and every tame character ξ, $\det\big(\mathcal{R}_j(\xi; W_j) \mid H_g(W_j)\big) = \det\big(\mathcal{R}_j(\xi; Y_0) \mid T\big)$.

Problems 57.4.1, 57.4.3, 57.4.5 and 57.4.6 prove the following.

Corollary 57.4.1. *Let $1 \le j \le m$. Then $\dim H_g(W_j) = d_T$; for every eigensystem λ the λ-eigenspaces of the $h_{w,i}$ in T and in $H_g(W_j)$ have the same dimension; and for every tame character ξ, $\mathcal{R}_j(\xi; Y_0)$ maps T to itself, $\mathcal{R}_j(\xi; W_j)$ maps $H_g(W_j)$ to itself, and $\det\big(\mathcal{R}_j(\xi; W_j) \mid H_g(W_j)\big) = \det\big(\mathcal{R}_j(\xi; Y_0) \mid T\big)$.*

The problems deduce Corollary 57.4.1 from Theorem 57.3.3, which rests on the comparison of trace formulas taken as given, and from facts read from the proof: finite-dimensionality under the gap hypothesis, the decomposition into joint eigenspaces, the series for the matrix coefficients of $\mathcal{R}_j(\xi; V)$, and the operators $\mathrm{av}(\gamma)$. Chapter 55 combines the corollary with Theorem 56.4.1, which writes $\hat{u}^*_{\xi,j}\hat{u}_{\xi,j}$ as a multiple of $\mathcal{R}_j(\xi; W_j)$, to express the volume of a single lift by a determinant on T (Problem 55.2.1).

In the Lean formalization. In the Lean proof [GL] (`grh/src/`, namespace `New.Data` unless written in full; EM and HE are the files `EqualMultiplicities.lean` and `HeckeEigensystems.lean` in `New/Spanning/Multiplicities/`), Theorem 57.3.3 is `trace_heckeOnGood_eq` (EM). Characteristic polynomials: `charpoly_heckeOnGood_eq`. The dimension assertion of Corollary 57.4.1 is `targetDim_eq_finrank_goodForms_spaceW`, for Y_0, W_j with $1 \le j \le m$; the multiplicity assertion specializes `finrank_goodForms_inf_eigenPart_eq`, stated for two spaces with the gap hypothesis; `rallisOnGood` restricts the Rallis operator to good forms on one, for any tame character and $j \in \mathbb{N}$, and `det_rallisOnGood_eq` equates the determinants on two, for the same character and j. All of these are in EM. Definition 57.3.1 is `New.HCParam.HasGap` for `paramW` and `gap`; the gap for Y_0, W_j is `hasGap_signsY_paramW`, `hasGap_paramW`, and finite-dimensionality the instance after `isCuspKernel_typeKernel_paramW`, in `New/UpperBound/Objects/Rows.lean`. Definition 57.3.2 uses `offLevelGroup`, `sphericalAlgebra`, `heckeAlgebra` and `heckeOperator` (HE). In HE the restriction to good forms is `heckeOnGood`, the eigenspaces are `eigenPart`, and their direct sum is given by `iSup_eigenPart_eq_top` (spanning) and `iSupIndep_eigenPart` (directness). The action of $\bar{G}$ is given by `coe_principalRep_apply` (HE) with `New.liftResidue_mem_supportedOn`, and commutation with it follows from `commute_heckeAlgebra_principalRep` (HE). The proof of `exists_hecke_proj_eigenPart` (HE) constructs the polynomial of Problem 57.4.2. At split places away from the level, `New.heckeSplit_adjoint` identifies the adjoint on principal forms with the operator at the conjugate place; these operators commute by `New.heckeSplit_comm`, which gives normality as well. The comparison of trace formulas is `New.trace_eq_of_isCuspidalAt` (`Lib/Adelic/TraceFormula/CuspidalComparison.lean`), with `New.IsLevel` and `New.IsCuspidalAt`; the reduction to it is `trace_heckeOnGood_eq_sum`, `exists_isPrimitivePolynomial` and `trace_mul_tupleIdem_eq` (EM), and the field conditions on the data follow from the instance of `New.HasPrimitiveRoot5040` for E in `New/Derivatives/Field/PrimeDecomposition.lean` and from

`f_even`. The series for $\mathcal{R}_j(\xi; V)$ is `hasSum_inner_rallisOperator` (`New/Determinant/Norms/AdjointThetaLift.lean`), its representatives and their countability `New.SiegelWeil.cosetRep` and `New.SiegelWeil.countable_quotient_localLevel`; the operators $\mathrm{av}(\gamma)$ and the common operator of Problem 57.4.5 are `exists_heckeAlgebra_inner_eq_inner_translate` and `exists_hecke_eq_rallisOperator` (EM).

Hints

§57.1.

57.1.1: $KgK = Kg \cup Kg(12)$; list the four permutations.

57.1.2: For $h \in K_g$, $g^{-1}hg \in K$ fixes y. Move δ to the other side of the inner product.

57.1.3: Write the operator as a 2×2 matrix; the traces of powers are sums of powers of its eigenvalues.

57.1.4: Take $h = 1$, then the traces of the powers of h, and Newton's identities.

57.1.6: Only finitely many λ occur on $U \oplus U'$.

§57.2.

57.2.1: Sage: the functions `BrandtModule` (with its Hecke matrices) and `EllipticCurve` (for the curve $X_0(11)$, whose traces of Frobenius are the $a_\ell(f)$). For $\ell = 2, 3$, Problems 24.4.2 and 24.4.3.

57.2.2: (b) The row sums of $B(\ell)$ and the relation between $B_{ij}(\ell)$ and $B_{ji}(\ell)$ quoted in Section 24.4; the constants are not in U_0.

57.2.3: Sage: the functions `BrandtModule` and `CuspForms`.

§57.3.

57.3.1: Problem 34.2.1 and its hint. For Y_0 the signs alternate along the entries after the semicolon. For W_j they change from $+1$ to -1 at index $k + j - 1$ if $j < m$, and all are $+1$ if $j = m$.

57.3.2: Compare components place by place; $e_0 \in \mathbb{C}[\bar{G}]$.

§57.4.

57.4.2: For each λ' choose (w, i) with $\lambda_{w,i} \neq \lambda'_{w,i}$; no commutation is needed.

57.4.3: Only finitely many eigensystems occur in $H(Y_0)$ or $H(W_j)$. Compare with Problem 57.1.6.

57.4.4: Finite-dimensional subspaces are closed.

57.4.5: Take K to be the group of integral elements of $G_f^{\mathcal{M}}$. It fixes the principal forms, and $K_\gamma := K \cap \gamma K \gamma^{-1}$ has finite index in K. For a choice of representatives of K/K_γ, the proof shows that the normalized sum of the $\delta\gamma$ in $\mathbb{C}[G_f^{\mathcal{M}}]$ is spherical; it induces the family $\mathrm{av}(\gamma)$, whose action on principal forms is the average of Problem 57.1.2. For the series, Problem 57.4.4 with the elements $\mathrm{mc}(\gamma)\,\mathrm{av}(\gamma)$.

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