THE NORM OF ONE LIFT

Prerequisites. Adjoints (inner products are conjugate-linear in the first argument). From Chapter 21: the class sets Cl_n, the groups Γ_L and Problems 21.1.2 and 21.1.4. From Chapter 23: the theta function $\Theta_{L,L'}$, the lift (1) and Problems 23.1.5, 23.5.1 and 23.3.2; the Gaussian constant $\vartheta_0 := \sum_{\alpha\in\mathbb{Z}[\mathrm{i}]} e^{-\pi|\alpha|^2}$ of that chapter. From Chapter 22: Section 22.4. From Chapter 34: $\mathcal{M}$, $\bar{G}$ (Section 34.1), W_j, V_j (Section 34.3), $H(V)$ (Section 34.4), with the Petersson product (Section 57.3), tame characters ξ, Q_ξ, (ψ_w) (Section 34.6), μ_j^0 (Section 34.7) and $\hat{u}_{\xi,j}$ on $H(W_j)$ (Section 34.9).

Theorem 55.4.1 of Chapter 55 expresses the volume of the chain, relative to the trivial twist, through the corresponding ratios for m single lifts on good forms, a power of the norm of the twist and ratios of L-values. For each it needs $\hat{u}^*\hat{u} =$ (scalar) $\cdot$ (Hecke series). How is the norm of a theta lift computed, and where does the twist enter?

56.1. The squared norm of a lift is a theta sum on the doubled space

We use the definite model of Chapter 23 in weight 0, from U(1) to U(3). By Problem 21.1.4 the class sets of I_1 and I_3 have one element each; let 1 be the constant form on Cl_1, so $\|1\|^2 = 1/4$. Write $\Theta_{(0)}\colon \mathcal{A}_{(0)}(1) \to \mathcal{A}_{(0)}(3)$ for the lift (1) in weight 0, and $\Theta'_{(0)}$ for the lift in the other direction. In weight 0 the kernel of (1) at (L, L') is the number $\Theta_{L,L'}(0) = \sum_{u\in \mathrm{Hom}_{\mathcal{O}}(L,L')} e^{-\pi h(u,u)}$.

Problem 56.1.1. $(\Theta_{(0)}1)(I_3) = \vartheta_0^3/4 = 0.4111137902\ldots$, where $\vartheta_0 = 1.1803405990\ldots$ is the square of $\sum_{n\in\mathbb{Z}} e^{-\pi n^2}$.

Problem 56.1.2.

$$\frac{\|\Theta_{(0)}1\|^2}{\|1\|^2} = \frac{\vartheta_0^6}{1536} = \frac{1}{384}\cdot\frac{1}{4}\sum_{(\alpha,\beta)\in\mathcal{O}^3\times\mathcal{O}^3} e^{-\pi(|\alpha|^2+|\beta|^2)} = 0.0017605682\ldots,$$

with $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$ and $|\alpha|^2 = \sum_k |\alpha_k|^2$.

The middle expression is a theta sum over $\mathrm{Hom}_{\mathcal{O}}(I_1 \oplus I_1, I_3) = \mathcal{O}^3 \times \mathcal{O}^3$; in general:

Problem 56.1.3. For unimodular Hermitian lattices L, L',

$$\Theta_{L,L'}(0)\,\overline{\Theta_{L,L'}(0)} = \Theta_{L\oplus L,L'}(0).$$

Hence, for the weight-0 lift $\Theta_{(0)}$ from Cl_n to $\mathrm{Cl}_{n'}$ and its reverse $\Theta'_{(0)}$, the kernel of $\Theta'_{(0)}\Theta_{(0)}$ at (L, L), $L \in \mathrm{Cl}_n$, is the sum of $|\Gamma_{L'}|^{-1}\,\Theta_{L\oplus L,L'}(0)$ over $L' \in \mathrm{Cl}_{n'}$.

The class sum models the adelic doubling integral: for a source $V^\flat$ and a target $V^\sharp$, a theta kernel times the conjugate of another is a theta function on $\mathrm{Hom}(V^\flat \oplus (V^\flat)^-, V^\sharp)$, and the doubled source ($(V^\flat)^-$ has the negated form) is split. Let

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V^0 and $V^\sharp$ be framed Hermitian spaces, in the sense of Section 34.3, over a CM field with a chosen CM type and Weil data, V^0 with a complete splitting into p hyperbolic planes (in this sheet p counts planes) and $V^\sharp$ of rank $n^\sharp > 2p$, and let ω be the split Weil action on the test space $\mathcal{S}$ of $\mathrm{Hom}(V^0, V^\sharp)$. For a rational Lagrangian $\mathcal{L}$ of V^0 let $D_{\mathcal{L}}(\phi)$ be the value at 0 of $\phi \in \mathcal{S}$ in the Schrödinger model of a splitting whose isotropic space is $\mathcal{L}$. Then the operator norms of the $D_{\mathcal{L}}$ are summable, $g^\sharp \mapsto \theta(\omega(1, g^\sharp)\phi)$ is integrable for every ϕ, and

$$\int_{[\mathrm{U}(V^\sharp)]} \theta(\omega(1, g^\sharp)\phi)\, dg^\sharp = \mathrm{vol}[\mathrm{U}(V^\sharp)] \sum_{\mathcal{L}} D_{\mathcal{L}}(\phi),$$

where $\mathrm{vol}[\mathrm{U}(V^\sharp)]$ is the total mass of the quotient measure. This is the Siegel–Weil formula in Weil's range [We65, Thm. 5]. *Status*: classical; the proof establishes it in this form. For $V^0 = V^\flat \oplus (V^\flat)^-$ the condition is $2n^\flat < n^\sharp$; for $n^\flat = 1$ it first holds for U(3), not for the U(2) of Problem 23.3.2.

For the summability the proof assigns to each $\mathcal{L}$ a vector $\mathbf{e} \in \mathbb{N}^p$ of dyadic exponents (with $0 \in \mathbb{N}$); with $|\mathbf{e}| := e_1 + \cdots + e_p$ and f the number of infinite places, at most $C\, 2^{2fp|\mathbf{e}|}$ Lagrangians have a given $\mathbf{e}$, and $\|D_{\mathcal{L}}\| \le C' 2^{-fn^\sharp|\mathbf{e}|}$.

Problem 56.1.4. For integers $p, f \ge 1$ and $n^\sharp$, $\sum_{\mathbf{e}\in\mathbb{N}^p} 2^{-f(n^\sharp-2p)|\mathbf{e}|} = \sum_{l\ge 0} \binom{l+p-1}{p-1} 2^{-f(n^\sharp-2p)l}$. It converges if and only if $n^\sharp > 2p$, with sum $(1 - 2^{-f(n^\sharp-2p)})^{-p}$; for $p = f = 1$, $n^\sharp = 3$ the sum is 2, and the terms with $l \le 5$ give $63/32$.

56.2. Only graphs survive, and unfolding

Which Lagrangians of the doubled space contribute? Over a finite field, let $\mathbb{F}_9 = \mathbb{F}_3[\mathrm{i}]$, with conjugation $z \mapsto \bar{z} := z^3$.

Problem 56.2.1. The norm $z \mapsto z\bar{z}$ from $\mathbb{F}_9$ to $\mathbb{F}_3$ has fibres of sizes $1, 4, 4$ over $0, 1, 2$. The form $z_1\bar{z}_1 - z_2\bar{z}_2$ on $\mathbb{F}_9^2$ has 33 isotropic vectors (including 0) and 4 isotropic lines, and these lines are the graphs $\{(\delta z, z) : z \in \mathbb{F}_9\}$ of the 4 elements δ of norm 1.

Problem 56.2.2. Let $V^\flat$ be a nondegenerate Hermitian space of dimension $n^\flat$ over a field with involution, with Gram matrix J, and give $V^\flat \oplus (V^\flat)^-$ the Gram matrix $\mathrm{diag}(J, -J)$. Let $\mathcal{L}$ be a Lagrangian, with frame $\binom{K_1}{K_2}$: a $2n^\flat \times n^\flat$ matrix of rank $n^\flat$ whose columns span $\mathcal{L}$. If K_2 is invertible, then $\delta := -K_1K_2^{-1}$ is an isometry of $V^\flat$ and $\mathcal{L} = \{(\delta z, -z) : z \in V^\flat\}$. If $V^\flat$ has no nonzero isotropic vector, every Lagrangian is of this form.

For the clean data ϕ of Theorem 56.2.1 below, the proof shows that only graphs of rational isometries contribute (by Problem 56.2.2 when $V^\flat$ is definite): for the doubled datum of the translates of ϕ', ϕ by g', g, the term of $\{(\delta^{-1}z, -z)\}$ is $\langle \phi', \omega(g'^{-1}\delta g)\phi\rangle$. Integrating against $y(g)\overline{y'(g')}$ then unfolds, as in this finite model.

Problem 56.2.3. Let Γ be a subgroup of a finite group G, let K be a function on G (in this sheet K is a function; the field $\mathbb{Q}(\sqrt{-d})$ does not occur), and let y, y' be functions on G invariant on the left under Γ, with $\langle y', y\rangle := \sum_{g\in\Gamma\backslash G} \overline{y'(g)}\, y(g)$. Then

$$\sum_{g,g'\in\Gamma\backslash G} \sum_{\delta\in\Gamma} K(g'^{-1}\delta g)\, y(g)\, \overline{y'(g')} = \sum_{g\in G} K(g)\, \langle y', y(\cdot\, g)\rangle.$$

The result is the Rallis inner product formula. Let $V^\flat$, $V^\sharp$ be framed Hermitian spaces over a CM field with a chosen CM type and Weil data, of ranks $n^\flat$, $n^\sharp$ with $2n^\flat < n^\sharp$, and let $G^\flat$ be the adelic unitary group of $V^\flat$. Let χ_1, χ_2 be continuous unitary characters of the norm-one ideles trivial on rational points, and let ϕ, ϕ' lie in the adelic test space of $\mathrm{Hom}(V^\flat, V^\sharp)$. Suppose that ϕ is *clean*: either all signs of $V^\flat$ agree, or $\phi = \phi_1 \otimes \phi_0$ for a Schwartz–Bruhat function ϕ_1 at a set of finite places disjoint from its conjugate such that, at some finite place, every matrix in the support of ϕ_1 has a left inverse. Let y, y' be automorphic functions on $G^\flat$ of rapid decay, let $\Theta[\phi]\, y$ be the theta lift of y to the group of $V^\sharp$ with datum ϕ and characters (χ_1, χ_2), and put $K(g) := \langle \phi', \omega(g)\phi \rangle \, \langle y', y(\cdot\, g) \rangle$, where $\omega(g)$ is the action of $(g, 1)$ with characters $(1, \chi_2)$ and the second factor is the Petersson product.

Theorem 56.2.1 (Rallis inner product formula)**.** *K is integrable on all of $G^\flat$, and $\langle \Theta[\phi']y', \Theta[\phi]y \rangle = \mathrm{vol}[\mathrm{U}(V^\sharp)] \int_{G^\flat} K(g)\, dg$.*

This is the unitary form, used here, of Rallis's inner product formula; for the classical method see [Ra87, §§ 2–3]. *Status*: classical; the proof derives it in this form from the Siegel–Weil formula, as Problems 56.2.2 and 56.2.3 model.

56.3. The sharp datum and the twist factor

For $1 \le j \le m$ the single lift $\hat{u}_{\xi,j}$ goes from W_j, of rank r, to V_j, of rank $r+t > 2r$; $H(W_j)$ is finite-dimensional (Chapter 57, Section 57.3 and Problem 57.3.1), so $\hat{u}_{\xi,j}$ has an adjoint. At the places of $\mathcal{M} \cup Q_\xi$ every residue matrix in the support of its finite datum has a left inverse; this support count and the count of all such matrices (through the target level index) give the finite factor in Theorem 56.4.1.

Problem 56.3.1. Of the 5^3 columns of height 3 over $\mathbb{F}_5$, exactly 124 have a left inverse; of the 5^6 columns of height 6, exactly 15624.

Problem 56.3.2. For integers $r \ge 1$ and $t \ge 0$, the number of $(r+t) \times r$ matrices over $\mathbb{F}_q$ with a left inverse is $q^{r(r+t)} \prod_{v=t+1}^{t+r} (1 - q^{-v})$.

Definition 56.3.1. For a finite set Q of finite places of E, with residue fields k_w of order q_w, the *twist factor* is

$$c_Q := (\mathrm{Nm}\, Q)^{rt} \prod_{w \in Q} \prod_{v=t+1}^{t+r} \left(1 - q_w^{-v}\right), \qquad \mathrm{Nm}\, Q := \prod_{w \in Q} q_w,$$

where r is the source rank and t the target rank minus the source rank; for the single lifts of the data, $t = 5r$ and $Q = Q_\xi$.

Problem 56.3.3. $c_Q = \prod_{w \in Q} (\lambda_w / q_w^{r^2})$, where λ_w (in this sheet) is the number of $(r+t) \times r$ matrices over k_w with a left inverse; in particular $c_\varnothing = 1$. Formally evaluating the expression of Definition 56.3.1 at $Q = \{w\}$, $q_w = 5$, $r = 1$ gives $c_Q = 124/5$ for $t = 2$ and $c_Q = 15624/5$ for $t = 5$.

Give the standard lattice stabilizer in each finite adelic unitary group Haar volume 1. When both lifts use the same sharp datum (below) and the unprimed form has principal level, the finite integration in Theorem 56.2.1 is a sum over cosets of the subgroup integral outside $\mathcal{M} \cup Q_\xi$ and their conjugates, and the integral on each coset supplies the mass of the finite datum as a factor; here is the finite model.

Problem 56.3.4. Let Γ and $\mathcal{J}$ be subgroups of a finite group G, let y, y' be functions on $\Gamma\backslash G$, and let y be invariant on the right under $\mathcal{J}$. With the pairing of Problem 56.2.3, $\sum_{g\in G} \mathbf{1}_{\mathcal{J}}(g)\,\langle y', y(\cdot\, g)\rangle = \#\mathcal{J}\,\langle y', y\rangle$.

The lift $\hat{u}_{\xi,j}$ uses a highest-vector Fock datum ϕ_∞ at the infinite places and a finite Weil vector F_{fin}, obtained from the vacuum by imposing, modulo the product of the places of $\mathcal{M}\cup Q_\xi$, the following residue function F (in this sheet) on families $\mathbf{X} = (X_w)_{w\in\mathcal{M}\cup Q_\xi}$ of $(r+t)\times r$ matrices over the k_w: $F(\mathbf{X}) := \prod_{w\in Q_\xi} \psi_w(\det X'_w)$ if X_w is the standard inclusion $\binom{1}{0}$ at every $w\in\mathcal{M}$, and $F(\mathbf{X}) := 0$ otherwise; here X'_w is the top $r\times r$ block of X_w, and $\psi_w(0) := 0$. Let $J_{\mathcal{M}}$ (in this sheet) be the principal level of $H(W_j)$ and K° the stabilizer of the standard lattice. Let S_0 be the set of places of $\mathcal{M}\cup Q_\xi$ and their conjugates. In the matrix coefficients of this lift take $\chi_2 = b_j := \mu_j^0\xi\,\upsilon^{\lfloor (r+t)/2\rfloor}$ in Theorem 56.2.1, with υ the splitting character on the norm-one ideles. The proof shows that F_{fin} is *sharp* for $J_{\mathcal{M}}$ off S_0: for every Fock vector v, $v\otimes F_{\mathrm{fin}}$ is clean and a vacuum off S_0; $\|F_{\mathrm{fin}}\|^2 = \langle F_{\mathrm{fin}}, F_{\mathrm{fin}}\rangle \neq 0$; and for finite g integral outside S_0, $\langle F_{\mathrm{fin}}, \omega(g)F_{\mathrm{fin}}\rangle = \|F_{\mathrm{fin}}\|^2\,\mathbf{1}_{J_{\mathcal{M}}}(g)$. The *mass* of F_{fin} is $\|F_{\mathrm{fin}}\|^2/[K^\circ : J_{\mathcal{M}}]$, where $\|F_{\mathrm{fin}}\|^2$ is the mean of $|F(\mathbf{X})|^2$ over all families $\mathbf{X}$ and $[K^\circ : J_{\mathcal{M}}] = \#\bar{G}$.

Problem 56.3.5. $\sum_{\mathbf{X}} |F(\mathbf{X})|^2 = \#\mathrm{GL}_r(Q_\xi)\cdot(\mathrm{Nm}\,Q_\xi)^{rt}$, where $\#\mathrm{GL}_r(Q) := \prod_{w\in Q}\#\mathrm{GL}_r(k_w)$.

The class-sum product on the target of $\hat{u}_{\xi,j}$ is the index of the level of V_j times the dimension of the coefficient representation times the Petersson product of the associated adelic forms, and the proof shows that this index times $\#\mathrm{GL}_r(Q_\xi)$ is $\prod_{w\in\mathcal{M}\cup Q_\xi}\lambda_w$, the size of the orbit of the standard inclusion. Put

$$c_{\mathrm{ch}} := \prod_{w\in\mathcal{M}}\left(\frac{1}{\#\mathrm{GL}_r(k_w)}\prod_{v=t+1}^{t+r}\left(1-q_w^{-v}\right)\right),$$

a nonzero number that does not depend on ξ.

Problem 56.3.6. With λ_w as in Problem 56.3.3,

$$\frac{(\mathrm{Nm}\,Q_\xi)^{rt}}{\#\bar{G}}\prod_{w\in\mathcal{M}\cup Q_\xi}\frac{\lambda_w}{q_w^{(r+t)r}} = c_{\mathrm{ch}}\,c_{Q_\xi}.$$

So the index of the level of V_j times the mass of F_{fin} is $c_{\mathrm{ch}}c_{Q_\xi}$, and only the factor c_{Q_ξ} depends on ξ.

At the other places the integral becomes a sum over Hecke cosets. Write A (in this sheet) for the set of finite places outside S_0. For a finite point γ of the unitary group of a framed space V of rank r let $\Delta_f(\gamma)$ be its finite Weil multiplier and (in this sheet)

$$\mathrm{mc}(\gamma) := \left(b_j(\det\gamma)\,\Delta_f(\gamma)^{r+t}\right)^{-1},$$

so $\mathrm{mc}(1) = 1$. For a datum that is a vacuum on A, left multiplication by γ supported on A multiplies its matrix coefficient at g, with g integral on A, by $\mathrm{mc}(\gamma)$.

Definition 56.3.2. Let $j\geq 0$ be an integer, ξ a tame character, and V a framed space of rank r, with any signs, such as W_j or Y_0. The *Rallis operator* $\mathcal{R}_j(\xi;V)$ is the endomorphism of $H(V)$ given by

$$\mathcal{R}_j(\xi;V)\,y := \sum_\gamma \mathrm{mc}(\gamma)\,y(\cdot\,\gamma),$$

where γ runs over representatives, equal to 1 outside A, of the finite points of $\mathrm{U}(V)$ modulo those integral at every place of A.

Through the frames, all framed spaces V of rank r have the same finite adelic unitary group, so one family of representatives γ serves every V; the representatives are 1 at the places of $\mathcal{M}$, and neither they nor the coefficients $\mathrm{mc}(\gamma)$ depend on V.

The proof shows that for $2r < r + t$ the series converges absolutely, lies in $H(V)$, and $\langle y', \mathcal{R}_j(\xi; V)\, y\rangle = \sum_\gamma \mathrm{mc}(\gamma)\langle y', y(\cdot\,\gamma)\rangle$ for $y, y' \in H(V)$; it proves the summability of the coefficients by counting neighbouring lattices. For $1 \le j \le m$, the sharp-data form of Theorem 56.2.1 and Problem 56.3.6 give, for every tame character ξ and all $y, y' \in H(W_j)$,

$$\langle \hat{u}_{\xi,j}\, y', \hat{u}_{\xi,j}\, y\rangle = a_j\, c_{Q_\xi}\, \langle y', \mathcal{R}_j(\xi; W_j)\, y\rangle,$$

where a_j is c_{ch} times the dimension of the coefficient representation times a constant c_∞ (in this sheet) that depends only on the spaces and the real data. As ξ has weight 0, a_j does not depend on ξ. Here $c_\infty = \mathrm{vol}[\mathrm{U}(V_j)]\, z_\infty$, where z_∞ (in this sheet) is the integral over the real source group of $\langle \phi_\infty, \Omega(x,1)\phi_\infty\rangle\, \overline{e(x)}/e(1)$, with e the source type kernel and Ω the real Weil action. The proof shows that $z_\infty \neq 0$ and that the lift with datum $\phi_\infty \otimes F_{\mathrm{fin}}$ takes forms of the source type to forms of the target type; for the correspondence at infinity see [Li90].

56.4. Injectivity and the theorem

Is $\mathcal{R}_j(\xi; V)$ injective? The proof partitions A into finitely many disjoint conjugation-stable sets, on each of which the neighbour weights (the number of steps at w times $q_w^{-(r+t)/2}$) add up to at most $1/3$. By sharpness, $b_j \circ \det$ is trivial on the integral points supported on A. On bounded functions fixed by the integral points supported on these places, the series over a union of parts is the composite of their series; on one part the moduli of the $\mathrm{mc}(\gamma)$ off the trivial coset add up to at most $1/2$, by a walk whose one-step weights total at most $1/3$.

Problem 56.4.1. $\sum_{l \ge 1} 3^{-l} = 1/2$. Let y be a bounded function on a group G, and let $g_i \in G$ and $\zeta_i \in \mathbb{C}$, $i \ge 1$, with $\sum_i |\zeta_i| \le 1/2$. If $y + \sum_i \zeta_i\, y(\cdot\, g_i) = 0$, then $y = 0$.

With Problem 56.4.1 this gives what the proof shows: $\mathcal{R}_j(\xi; V)$ is injective.

Problem 56.4.2. Let $u\colon \mathcal{E} \to \mathcal{E}'$ be a linear map of finite-dimensional complex inner product spaces, $\mathcal{R}$ an injective endomorphism of $\mathcal{E}$, and $\zeta \in \mathbb{C}$, $\zeta \neq 0$. If $\langle uy', uy\rangle = \zeta\langle y', \mathcal{R}y\rangle$ for all $y, y' \in \mathcal{E}$, then u is injective.

Problem 56.4.3. In Problem 56.4.2, the identity $\langle uy', uy\rangle = \zeta\langle y', \mathcal{R}y\rangle$ for all y, y' is equivalent to $u^*u = \zeta\,\mathcal{R}$.

Problems 56.3.4–56.3.6 compute the finite factor $c_{\mathrm{ch}} c_{Q_\xi}$ of the Gram identity of Section 56.3, whose other inputs we took from the proof, and Problem 56.4.3 turns it into the following theorem, which the proof states in this form.

Theorem 56.4.1. *For every $1 \le j \le m$ there is $a_j \in \mathbb{C}$, $a_j \neq 0$, such that for every tame character ξ,*

$$\hat{u}^*_{\xi,j}\, \hat{u}_{\xi,j} = a_j\, c_{Q_\xi}\, \mathcal{R}_j(\xi; W_j)$$

on $H(W_j)$.

By Problem 56.4.2 and the injectivity of $\mathcal{R}_j(\xi; W_j)$, the lift $\hat{u}_{\xi,j}$ is injective.

Taken as given were the Siegel–Weil formula in Weil's range, Theorem 56.2.1 in its sharp form, the summability of the coefficients, the archimedean properties of ϕ_∞, the sharpness of F_{fin}, the class-sum product and the index identity on the target of $\hat{u}_{\xi,j}$, and the change of matrix coefficients under left multiplication, and the bound on the neighbour weights behind the injectivity of $\mathcal{R}_j(\xi; V)$. Chapter 57 compares $\det \mathcal{R}_j(\xi; W_j)$ and $\det \mathcal{R}_j(\xi; Y_0)$ on good forms. The rank hypothesis of Theorem 56.2.1 holds for $Y_0 \to Y_1$ but not for $Y_1 \to Y_2$, as $2(r+t) > r + 2t$; the chain identity uses the exchange identity of Chapter 58 instead.

In the Lean formalization. In the Lean proof [GL], Theorem 56.4.1 is `New.Data.exists_adjoint_singleLift_comp_eq` in `New/Determinant/Norms/AdjointThetaLift.lean`, and the Gram identity of Section 56.3, with $\langle y', \mathcal{R}_j(\xi; W_j)\, y\rangle$ written as its series over γ, is `Data.exists_hasSum_inner_singleLift` there. See also, in that file, `DescentPair.twistFactor`, `Data.rallisOperator`, `DescentPair.isSharp_blockDatum` and `DescentPair.gramFactor_div_card`, and in `Lib/SiegelWeil/` the declarations `New.SiegelWeil.siegel_weil`, `New.SiegelWeil.inner_product_formula` (Theorem 56.2.1), `New.SiegelWeil.rallis_inner_product_cosetTerm` and `New.SiegelWeil.eq_zero_of_rallisSeries_eq_zero`.

Hints

§56.1.

56.1.1: Equation (1) with $|\Gamma_{I_1}| = 4$; $\mathrm{Hom}_{\mathcal{O}}(I_1, I_3) = \mathcal{O}^3$, and the sum over $\mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}\mathrm{i}$ factors.

56.1.2: The squared norm on $\mathcal{A}_{(0)}(3)$ and Problem 21.1.2 for I_3.

56.1.3: $\mathrm{Hom}_{\mathcal{O}}(L \oplus L, L') = \mathrm{Hom}_{\mathcal{O}}(L, L')^2$; Problems 23.5.1 and 23.1.5.

56.1.4: Count the $\mathbf{e} \in \mathbb{N}^p$ with $|\mathbf{e}| = l$, and expand $(1-z)^{-p}$.

§56.2.

56.2.1: $(u_1 + u_2\mathrm{i})\overline{(u_1 + u_2\mathrm{i})} = u_1^2 + u_2^2$.

56.2.2: Isotropy reads $K_1^* J K_1 = K_2^* J K_2$; then look at vectors of $\mathcal{L}$ of the form $(z, 0)$.

56.2.3: Fix representatives of $\Gamma\backslash G$ and the representative g'. As δ and the representative g vary together, which values does $g'^{-1}\delta g$ take?

§56.3.

56.3.2: Left inverse means independent columns; choose them one at a time.

56.3.5: Sum place by place; $|\psi_w(d)| = 1$ for $d \neq 0$.

56.3.6: Problem 56.3.2, and the definition of $\bar{G}$.

§56.4.

56.4.1: Compare $\sup|y|$ with its half.

References

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