THE SIZE OF THE COMPOSITE LIFT

Prerequisites. Stirling's formula; exterior powers and the singular value decomposition of a complex matrix [HJ, §2.6]. From Chapter 33: the data (Definition 33.3.1), the weights κ, the prime sums s_{fin}, the sizes s_∞, Ω, h_{main}, s_{main}, rem and U (Section 33.4), and the level primes (Section 33.5). From Chapter 34: the places Φ, Φ_a, the spaces X_j, Y_j, the source forms S, whose values lie in Weyl modules with their Weyl (Gram) norms, the lift weights κ_j, the gap $\Delta = 2r^8$, and the diagram $\beta(\tau, h)$ (Definition 34.2.1). From Chapter 48, Section 48.1: c_p, I_p, $s_{\mathrm{fin},p}$ and gain(p); and weak chain data (Definition 48.3.2), the restricted chain $R_{\vartheta,\Psi}$ (Definition 48.3.3) and the expansion of a chain datum into a weak one (Theorem 48.3.4). From Chapter 43, Section 43.4: translations $\mathfrak{t}$, translated vacua $v_j(\mathfrak{t})$, chain data and the composite lift $B_{\vartheta,\Psi}$ with its restriction to Y_m. From Chapter 50: the jet scale c_{jet} (Definition 50.4.3) and the kernel of one lift (Section 50.4). The Chowla–Selberg formula, Theorem 20.3.1; the Fock norm $\|z^a\|^2 = a!$ of Problem 5.4.4; Chapter 23 for the smallest definite lifts.

Theorem A (Chapter 42) compares the size of the composite lift with its divisibility. Chapter 48 treats the divisibility; this sheet treats the size. How large can $B_{1,\Psi}\varphi$ be on every conjugate, if φ is small? The answer has two halves. At one pair of points the kernel of lift j multiplies conjugate Weyl norms by at most $\exp(x^{50} + \mathrm{rs}_j/2)$, where rs_j sums $s_\infty(l)$ over its rows at all places (Section 51.2): a Gaussian integral, and one factorial $l!$ per row of length l. The finite prime sums cancel exactly against the gains (Problem 51.1.4), and at each long place the level N pays for the excess of $m\, s_\infty(a)$ over $m\, s_{\mathrm{main}}$ up to $32a \log b$ (Problem 51.1.7).

51.1. The exponent

We recall the global prime sums and sizes from Chapter 33 and the local gains from Chapter 48. The places Φ are identified with $(\mathbb{Z}/q_1)^\times$, $[\nu]$ is the place of the residue class of an integer ν if $\gcd(\nu, q_1) = 1$, and the identity place otherwise, and Φ_a is the set of a-places. Recall $s_\infty(j) = \log j! + j \log(M/\Omega)$ with $\Omega = e^{\mathrm{CS}(d)}\, 2\pi/\mathrm{rd}_E$, $c_1 = -1 - \mathrm{CS}(d)$, $s_{\mathrm{main}} = a(\log a - \frac{1}{f}\sum_{\tau\in\Phi} \log \kappa(\tau) + c_1)$, $h_{\mathrm{main}} = mk \sum_{\tau\in\Phi}(s_\infty(\kappa(\tau)) - s_{\mathrm{fin}}(\kappa(\tau), \tau))$ and $\mathrm{rem} = \sum_{\tau\in\Phi_a}(s_{\mathrm{fin}}(a, \tau) - s_{\mathrm{main}})$. Let Π be the set of primes $p < x_1$ with $\chi_K(p) \neq 1$ (Section 48.1), and put, in Part VIII,

$$\Sigma = \sum_{p\in\Pi} \mathrm{gain}(p) \log p.$$

We use the following consequences of Chapter 33 (Definition 33.3.1 and Problems 33.2.3, 33.4.1, 33.4.4 and 33.5.4): $x_1 \le a$; $a/x_1 \le \kappa(\tau) \le b \le a$ for every τ; $20 \log b \le \log(M/\Omega) \le 23 \log b$ (Problem 33.4.3); $s_\infty(j) \ge 0$ for integers $0 \le j \le a$; $\log N \ge L_-(\kappa, m, M, \log b)$, as in Definition 33.3.1; $m(1 + \frac{1}{2}\log a) \le a \log b$; and $\log(M \mathrm{rd}_E/2\pi) \le 23 \log b$.

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Problem 51.1.1. For a prime power $\nu = p^e$, $e \geq 1$, one has $\Lambda(\nu)(1 - \chi_K(\nu)) = 2c_p \log p$ if $e \in I_p$, and 0 otherwise.

Problem 51.1.2. For every integer $P_0 \geq \lceil x_1 \rceil$, real $y \leq a$ and $\sigma \in \Phi$, $s_{\mathrm{fin}}(y, \sigma) = \sum_{p<P_0} 2\, s_{\mathrm{fin},p}(y, \sigma) \log p$, the sum over primes; in particular one may sum over $p < x_1$.

Problem 51.1.3. If $\chi_K(p) = 1$ or $p \geq x_1$, then $\mathrm{gain}(p) = 0$. Consequently

$$2\Sigma = m \sum_{\tau \in \Phi_a} s_{\mathrm{fin}}(a, \tau) + mk \sum_{\tau \in \Phi} s_{\mathrm{fin}}(\kappa(\tau), \tau).$$

Problem 51.1.4. $h_{\mathrm{main}} - m\,\mathrm{rem} + 2\Sigma = m\,\#\Phi_a\, s_{\mathrm{main}} + mk \sum_{\tau \in \Phi} s_\infty(\kappa(\tau))$.

Problems 51.1.2 and 51.1.3 recall Problems 48.1.4 and 48.1.3 in the form used here. The prime sums have cancelled.

Problem 51.1.5. For every integer $n \geq 1$, $\log n! \leq n \log n - n + 1 + \frac{1}{2} \log n$.

Problem 51.1.6. $\log(M/\Omega) = 1 + c_1 + \log(M \mathrm{rd}_E / 2\pi)$, and

$$s_\infty(a) - s_{\mathrm{main}} \leq a\Big(\frac{1}{f} \sum_{\tau \in \Phi} \log \kappa(\tau) + \log \frac{M \mathrm{rd}_E}{2\pi}\Big) + 1 + \tfrac{1}{2} \log a.$$

Problem 51.1.7. $m(s_\infty(a) - s_{\mathrm{main}}) - a \log N \leq 32\, a \log b$.

Put $C_{\mathrm{ch}} = m\big(\#\Phi_a\, s_\infty(a) + k \sum_{\tau \in \Phi} s_\infty(\kappa(\tau))\big)$.

Problem 51.1.8. $C_{\mathrm{ch}} - \#\Phi_a\, a \log N \leq h_{\mathrm{main}} - m\,\mathrm{rem} + 2\Sigma + 32\, f a \log b$.

For $n = 6$ the two sides of Problem 51.1.5 compare as follows.

Problem 51.1.9. $\log 6! + 6(1 - \log 6) = 1.8286\ldots \leq 1 + \frac{1}{2} \log 6 = 1.8958\ldots$.

51.2. One kernel in one coordinate

Where does s_∞ come from? In the proof the conjugate of a theta jet of a moved translate of the integral vacuum is written as a Gaussian integral of a conjugated polynomial against the vacuum $e^{-\sum_i |y_i|^2/2}$ and a bounded function [GL]. In this sheet l denotes the length of a row, $L \geq 1$ a growth rate, and v, t_0, u are real numbers.

Problem 51.2.1. For $\rho \geq 0$, $l \in \mathbb{N}$ and $v \in \mathbb{R}$, $\rho^l \leq l!\, \exp(lv + \rho e^{-v})$.

Problem 51.2.2. In the one-coordinate model let $l \geq 0$ be an integer, $t_0 > 0$, $L \geq 1$ and $u \geq 0$. If $z \in \mathbb{C}$ and $|z|^2 \leq t_0(2L + 2)u$, then

$$|z|^l \leq \exp\Big(\tfrac{1}{2}\big(l/L + \log l! + l \log t_0\big) + Lu\Big).$$

For a matrix the same device is applied to each row of a diagram. In one variable the Fock norm $\|z^l\|^2 = l!$ of Problem 5.4.4 shows the same factorial.

Problem 51.2.3. Let $n \geq 1$ and let Z be a complex $n \times n$ matrix and $\rho_0 \geq \rho_1 \geq \cdots \geq \rho_{n-1}$ the eigenvalues of Z^*Z. For $1 \leq h \leq n$ the operator norm of $\wedge^h Z$, for the inner product induced on $\wedge^h \mathbb{C}^n$, satisfies $\|\wedge^h Z\|^2 = \rho_0 \rho_1 \cdots \rho_{h-1}$. For $l \geq 1$, with the inner product on $\mathrm{Sym}^l \mathbb{C}^n \subseteq (\mathbb{C}^n)^{\otimes l}$ induced from the tensor power, $\|\mathrm{Sym}^l Z\|^2 \leq \rho_0^l$.

A column of height h carries $\wedge^h$, a row of length l carries Sym^l; a general diagram mixes both, and the proof uses the following bound. *Schur norm.* For each τ in a finite set let z_τ be a complex $n'_\tau \times n_\tau$ matrix with $n_\tau \le n'_\tau$, let $b_{\tau,h}$ be numbers of columns of height h, with $b_{\tau,h} = 0$ for $n_\tau < h < n'_\tau$, and put $l_{\tau,i} = \sum_{i<h<n'_\tau} b_{\tau,h}$. There are $\rho_{\tau,i} \ge 0$, $0 \le i < n_\tau$, with $\sum_{i<n_\tau} \rho_{\tau,i}$ at most the sum of the squared absolute values of the entries of z_τ for each τ, such that the Schur matrix of (z_τ) multiplies the squared Weyl norm of every vector by at most $\prod_\tau \prod_{i<n_\tau} \rho_{\tau,i}^{l_{\tau,i}}$. *Status*: classical linear algebra (singular values); read from the proof [GL].

Problem 51.2.4. Let $L \ge 1$ and $s' \in \mathbb{R}$. Then $Ls'/(2L+2) - s'/2 = -s'/(2L+2)$, and $\frac{1}{\pi}\int_{\mathbb{C}} e^{-|y|^2/c}\,dA(y) = c$ for $c > 0$. For $L = 2$, $\int_0^\infty 2re^{-r^2/6}\,dr = 6 = 2L+2$.

Problem 51.2.5. Let $l \ge 0$ be an integer, $t_0 > 0$, $L \ge 1$, $\mathrm{gr}(l) = \frac{1}{2}(l/L + \log l! + l\log t_0)$, and $H\colon \mathbb{C} \to \mathbb{C}$ measurable with $|H| \le 1$. Then

$$\Big|\frac{1}{\pi}\int_{\mathbb{C}} (\sqrt{t_0}\,y)^l e^{-|y|^2/2} H(y)\,dA(y)\Big| \le e^{\mathrm{gr}(l)}(2L+2),$$

and the Gaussian $\exp(-\sum_i |y_i|^2/(2L+2))$ in n_{c} complex coordinates has integral $(2L+2)^{n_{\mathrm{c}}}$ for the measure $\pi^{-n_{\mathrm{c}}} dA$.

The lift takes its Taylor series at the scale $\pi_M c_{\mathrm{jet}}$, with c_{jet} the jet scale of Definition 50.4.3. For each σ over K the conjugation law below gives a comparison scale t_σ with $|t_\sigma|^2 = 1/\Omega$, and $|\sigma(\pi_M)|^2 = M$; so the growth parameter is $t_0 = |\sigma(\pi_M)t_\sigma|^2 = M/\Omega$, and $2\,\mathrm{gr}(l) = l/L + s_\infty(l)$. Every automorphism of $\mathbb{C}$ gives the same absolute values as one over K [GL]. With $L = a+1$, $l/L \le 2$ for $l \le 2a$. The computation of $|t_\sigma|^2$ uses the area of the conjugate lattice, the Chowla–Selberg unit and the normalization of the vacuum [GL]; the next two inputs enter it.

Conjugates of translated theta functions. Let $d \equiv 7 \pmod 8$ be the prime of the data. A *normalized CM lattice* is a lattice $\Lambda \subset \mathbb{C}$ stable under $\mathcal{O}_K$, with a point $t' \in \mathbb{C}$ of exact order two modulo Λ such that $(1 \pm \sqrt{-d})t'/2 \notin \Lambda$, scaled so that $\wp''_\Lambda(t') = 2$, with a chosen a_1 such that $a_1^2 = 12\wp_\Lambda(t')$; for a lattice D write $T_v^D f(z) = \exp(-\pi(z\bar v + |v|^2/2)/\operatorname{area}(D))f(z+v)$, and let θ be the normalized theta section of Λ of [GL] (the twisted translate by t' of the lattice sigma function, divided by its value at 0). Suppose $\Lambda = \omega_0\mathcal{O}_K$, and let σ be an automorphism of $\mathbb{C}$ over K. There are $\omega' \in \mathbb{C}^\times$, a nonzero ideal $\mathfrak{a}$ of $\mathcal{O}_K$ and a finite adele $\mathfrak{b}$ of K such that the lattice Λ' of the conjugate curve is $\omega'\mathfrak{a}^2/\mathrm{Nm}\,\mathfrak{a}$, has area $|\omega'|^2\sqrt{d}/2$ and contains $n_0\omega'$ for some integer $n_0 \ge 1$; and, with θ' defined in the same way from the conjugate marked lattice, for every $y \in K$ and every $y' \in K$ close enough to $\mathfrak{b}y$ in the finite adeles, applying σ coefficientwise to the Taylor series at 0 of $T^\Lambda_{y\omega_0}\theta$ gives the Taylor series at 0 of $T^{\Lambda'}_{y'\omega'}\theta'$. *Status*: this theta formulation is read from the proof [GL]; for the classical main theorem of complex multiplication see [Sh, Thm. 5.4].

The Chowla–Selberg unit. For every normalized CM lattice for K, in any ideal class, there is $u_{\mathrm{CS}} \in \mathbb{C}$ with $|\eta(u_{\mathrm{CS}})|_p = 1$ for every prime p and every ring map $\eta\colon \mathbb{C} \to \mathbb{C}_p$, such that for every automorphism σ of $\mathbb{C}$ over K the area of the lattice of the conjugate curve times $|\sigma(u_{\mathrm{CS}})|^2$ is $\pi e^{\mathrm{CS}(d)}\,2\pi/\sqrt{d}$. *Status*: Theorem 20.3.1 is the classical Chowla–Selberg product over ideal classes [CS]; the statement above is read from the proof [GL].

Let β be the diagram of Definition 34.2.1. Put $a_\tau = a$ if $\tau \in \Phi_a$ and $a_\tau = 0$ otherwise.

Definition 51.2.6. Fix $j < m$. For $\tau \in \Phi$ and $0 \le i < n_j$ let

$$\mathrm{row}_\tau(i) = \sum_{i<h<n_{j+1}} \beta(\tau, h),$$

the length of row i, counted from 0, of the diagram at τ formed by the columns of height less than n_{j+1}. We call these the *row lengths of lift* j, drop j from the notation, and put $\mathrm{rs}_j = \sum_{\tau\in\Phi} \sum_{0\le i<n_j} s_\infty(\mathrm{row}_\tau(i))$.

Write $G_f(V)$ for the group of finite adelic points of the unitary group of a space V. For $j < m$, a translation $\mathfrak{t}$, $g \in G_f(X_j)$ and $g' \in G_f(X_{j+1})$, consider the kernel of lift j of Section 50.4, at the trivial twist and at the datum $v_j(\mathfrak{t})$ moved by (g, g'): a linear map from the Weyl vectors of the weight of X_j to those of X_{j+1}, in which the values of forms lie. The proof establishes [GL]: *on every conjugate this kernel multiplies Weyl norms by at most* $\exp(x^{50} + \mathrm{rs}_j/2)$; that is, for every value ξ and every $\sigma \in \mathrm{Aut}(\mathbb{C})$, the Weyl norm of σ applied to the image of ξ is at most $\exp(x^{50} + \mathrm{rs}_j/2)$ times the Weyl norm of $\sigma\xi$. Here $L = a + 1$, $t_0 = M/\Omega$ and there are $n_{\mathrm{c}} = f n_j n_{j+1}$ coordinates; besides $\mathrm{rs}_j/2$ the exponent contains the logarithm of the vacuum bound, $n_{\mathrm{c}} \log(2a + 4)$ and $\sum_{\tau,i} \mathrm{row}_\tau(i)/(2L)$, and their sum is at most x^{50}.

51.3. The chain

The kernel bounds multiply along each tuple of classes, so their exponents involve $\sum_{j<m} \mathrm{rs}_j$. We bound this sum, count the classes, and restrict to Y_m.

Problem 51.3.1. For integers $l, l_0, l_1 \ge 0$ with $l \le l_0 + l_1$, $s_\infty(l) \le s_\infty(l_0) + l_1\big(\log(l_0 + l_1 + 1) + \log(M/\Omega)\big)$.

Problem 51.3.2. Let $j < m$, $\tau \in \Phi$ and $0 \le i < n_j$. Then $\mathrm{row}_\tau(0) \le a_\tau + b + r^{16}$ (the first-row slack in the proof is $b + a/x_1 + r^{16}$), $\mathrm{row}_\tau(i) \le \kappa(\tau) + r^{16}$ for $0 < i < k$, and $\mathrm{row}_\tau(i) \le r^{16}$ for $i \ge k$. Each row length is at most $2a$.

Problem 51.3.3. $\sum_{j<m} \mathrm{rs}_j \le C_{\mathrm{ch}} + U/2$.

Class count. Let E be a CM field with a chosen CM type and f infinite places, N' a positive integer in the different of E, and r' a nonnegative integer. The standard definite hermitian space of rank r' has at most $(1 + 2c_{\mathrm{sb}}^3)^{4fr'^2}$ classes $\Gamma\backslash G_f/K^\circ$, with Γ the rational points and K° the stabilizer of the standard lattice, where $c_{\mathrm{sb}} = 6N'(fr')^2$; the proof uses short bases in the standard space and applies the bound with $N' = q_1 d$; for sharper transference bounds see [Ba]. For the data, at a level containing the principal congruence subgroup of $\pi_M \pi_N$, in rank at most n_m, there are at most $e^{x^{30}}$ classes. *Status*: read from the proof [GL].

Branching norm. The restriction from X_m to Y_m of Section 43.4 applies an integral branching matrix; its squared operator norm for the Weyl norms is at most $e^{x^{70}}$. *Status*: read from the proof [GL].

For a form F on X_j or on Y_m and a ring homomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$, $|\sigma F(g)|^2$ denotes the Weyl squared norm of the conjugate value.

Problem 51.3.4. For a form F on X_m, a ring homomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$ and $h \in G_f(Y_m)$, the restriction of F to Y_m satisfies

$$|\sigma(\mathrm{res}\, F)(h)|^2 \le \exp\Big(x^{70} - \log N \sum_{\tau\in\Phi} (a_\tau - m r^8)\Big)\, |\sigma F(\mathrm{diag}(h, 1))|^2,$$

and $\sum_{\tau\in\Phi}(a_\tau - mr^8) = \#\Phi_a\, a - fmr^8$. Formally, for one place with $a_\tau - mr^8 = 1$ and $\pi_N = 1 + 2\sqrt{-1}$ (so $N = 5$, which is not the level of any data), the scalar contributes $|1/\bar{\pi}_N|^2 = 1/5$ to the squared norm.

Problem 51.3.5. $x^{70} + fmr^8 \log N + 2(mx^{30} + x^{60} + mx^{50}) + 32fa \log b \le U/2$.

Let Ψ be a chain datum for the trivial twist. By Theorem 48.3.4, choose a weak chain datum Ψ' with $B_{1,\Psi} = R_{1,\Psi'}$ on S; then $\sum_{\mathfrak{t}} |\sigma(\Psi'(\mathfrak{t}))| \le e^{x^{60}}$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$. Fix $\varphi \in S$ and $y \in G_f(Y_m)$. Write $C_{\mathfrak{t}}\varphi$ for the chain of the m lifts of φ with the data $v_j(\mathfrak{t})$. Before restriction the value at $\mathrm{diag}(y, 1)$ is $\sum_{\mathfrak{t}} \Psi'(\mathfrak{t})(C_{\mathfrak{t}}\varphi)(\mathrm{diag}(y, 1))$, and the proof expands it as a sum over the $\mathfrak{t}$ in the support of Ψ' and over the tuples of classes at the levels of $X_0, \dots, X_{m-1}$, of $\Psi'(\mathfrak{t})$ times products of m kernels applied to values of φ. There are at most $e^{mx^{30}}$ tuples of classes. The resulting estimate for the restricted chain, whose exponent is controlled by Problems 51.1.8, 51.3.3, 51.3.4 and 51.3.5, gives the following theorem, which the proof establishes [GL].

Theorem 51.3.6. *Let Ψ be a chain datum for the trivial twist, $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\varphi \in S$ and $c \in \mathbb{R}$. If $|\sigma\varphi(g)|^2 \le e^c$ for every $g \in G_f(X_0)$, then for every $y \in G_f(Y_m)$*

$$|\sigma(B_{1,\Psi}\varphi)(y)|^2 \le \exp\bigl(c + h_{\mathrm{main}} - m\,\mathrm{rem} + 2\Sigma + U\bigr).$$

The theorem is a statement at the trivial twist only. The problems give the arithmetic estimates in its exponent; taken as given are the kernel bound with the Schur norm, the conjugation of translated theta functions and the Chowla–Selberg unit behind it, the class count, the branching norm, and the expansion of a chain datum. In the exponent Σ has the coefficient 2; Chapter 42 sets the divisibility of Chapter 48 against it in Theorem 42.3.2.

In the Lean formalization. In the Lean proof [GL] (prefix `New.` omitted), Theorem 51.3.6 is `Data.weylNormSq_compositeLift_le`, deduced through `Data.IsChainDatum.exists_isWeakChainDatum_compositeLift_eq` from the bound `Data.weylNormSq_restrictedChain_le` for the restricted chain, whose proof expands the chain by `Data.chainCombination_eq_sum`. The conjugate jet is written as an integral in `Parameters.map_jet_eq_integral` (after `Parameters.jet_jetSeries_mul`) and bounded in `Parameters.norm_jet_translate_le`, with the Taylor scale $\pi_M c_{\mathrm{jet}}$ of `Data.liftKernel`, the comparison scale from `Parameters.exists_isJetConjugation`, whose squared modulus $1/\Omega$ is computed in `Parameters.norm_gal_periodFactor_div_mul_areaScale_sq` inside `Parameters.exists_isJetConjugate_translates`, the reduction to automorphisms over K in `Parameters.exists_algEquiv_norm_eq`, $|\sigma(\pi_M)|^2 = M$ from `Data.norm_map_star_piMBar_sq`, and the growth from `sqrt_prod_pow_le_exp`; the first-row slack is in `Data.diagramWeight_le_rowMainLength_add_rowSlack`; the restriction estimate is `Data.weylNormSq_restrictToY_le`; the kernel bound of Section 51.2 is `Data.sqrt_weylNormSq_jetMap_le`, with rs_j the definition `Data.rowSizeSum`. The results taken as inputs on this sheet are `CMLattice.complex_multiplication_theta`, `CMLattice.chowla_selberg`, `weylNormSq_schur_le`, `Space.card_stdClasses_le`, applied with $N' = q_1 d$ in `Data.card_stdClasses_le_exp_mul_log`, with `Data.card_stdClasses_le_exp`, and `exists_branching_matrix` with `Data.isBranchingMap_branching`.

Hints

§51.1.

51.1.2: Problem 51.1.1; the terms with $p^e \geq x_1$ vanish because $\kappa \geq a/x_1$.

51.1.3: Consider I_p in the split case and the positive part in the large-prime case. Which two choices of y give the sums in gain(p)?

51.1.4: Problem 51.1.3.

51.1.5: Show that $\log n! - (n+\frac{1}{2})\log n + n$ decreases in n; the series of $\frac{1}{2}\log\frac{1+t}{1-t}$ helps.

51.1.6: Problem 51.1.5 with $n = a$.

51.1.7: Compare Problem 51.1.6 with the lower bound for $\log N$. Which terms of size $ma(\frac{1}{f}\sum \log\kappa + \log\frac{M\mathrm{rd}_E}{2\pi})$ cancel?

51.1.8: Problems 51.1.4 and 51.1.7, and $\#\Phi_a \leq f$.

§51.2.

51.2.1: One term of the exponential series.

51.2.2: Apply Problem 51.2.1 to $\rho = |z|^2$ with v independent of z; which v makes $(2L+2)t_0 e^{-v} \leq 2L$?

51.2.3: Singular value decomposition $Z = PDQ^*$; $\wedge^h$ and Sym^l of a unitary matrix are unitary.

51.2.4: Polar coordinates.

51.2.5: Problem 51.2.2 with $u = |y|^2/(2L+2)$, then Problem 51.2.4.

§51.3.

51.3.1: Compare $l!$ with $(l_0+l_1)!$; what is the sign of $\log(M/\Omega)$?

51.3.2: Definition 34.2.1: which heights below n_{j+1} carry columns, and how many? Use $\kappa_1(\tau) = \kappa(\tau) + \frac{3}{2}\Delta$, $\kappa(\tau) \leq b$ and the sizes of m, r, x, b in Definition 33.3.1.

51.3.3: Problem 51.3.1 with $l_0 = a_\tau$ in row 0, $l_0 = \kappa(\tau)$ for $0 < i < k$ and $l_0 = 0$ otherwise; how much can one extra box cost (Problem 33.4.3)? Note $k - 1 \leq k$. Then $s_\infty(0) = 0$, $s_\infty(\kappa) \geq 0$, and $m \leq 2\sqrt{x}$, $a = xb$.

51.3.4: The scalar of the restriction; $|\psi(\bar{\pi}_N)|^2 = N$ for every embedding $\psi\colon E \to \mathbb{C}$, since $\pi_N\bar{\pi}_N = N$ and every complex embedding of E respects complex conjugation. An integral matrix commutes with σ.

51.3.5: Problem 33.5.4 for $\log N$; compare each term with $fa\log^3 b$, using $b \geq x_1^{10^6}$.

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