ONE LIFT AT A SMALL NON-SPLIT PRIME

Prerequisites. Chapter 33: the scales, ranks and weights, in particular $x_1 \mid a$ and $a/x_1 \le \kappa(\tau) \le b$ (Problem 33.4.1), and that x is very large. Chapter 34: the gap Δ, the weights κ_j (Section 34.2), the integral vacuum (Section 34.8), the diagram of the data (Definition 34.2.1), and the valuation val and the ring $\mathcal{O}_\eta$ of its opening paragraph. Chapter 48: c_p, I_p, $s_{\mathrm{fin},p}$, gain(p) (Section 48.1), collisions and $C_p(\Phi')$ (Section 48.2), the gauge $\|\omega\|_g$ of a Weyl vector and $\gamma_p = 1/(p-1) - \rho(d,p)$ (Section 48.3). Chapter 49: diagrams, β_1, $\beta_{>1}$, $\|\beta\|_{>1}$, $|\beta|$ (Definition 49.2.1), the erased diagram $\beta \smallsetminus \Phi'$ (Definition 49.2.2) and the predicted divisibility $\mathcal{D}(\beta)$ (Definition 49.2.3). Chapter 17, Section 17.5: the exponent set I_p°, couplings, the normal form, standard fillings, products of minors P_μ, and Theorem 17.5.1. Chapter 43: the definite chain with its data (Section 43.4); we write Heis(y) for the finite Heisenberg translation by y. The residue field $\bar{\mathbb{F}}_p$ of $\mathcal{O}_{\mathbb{C}_p}$, which is infinite.

How divisible by p is a single lift of the chain, and where does it fall short of gain$(p)/m$? Chapter 48 composes m lifts and needs, at each stage, a bound for the kernel of the lift at a prime $p < x_1$ with $\chi_K(p) \neq 1$ (its inputs (ii) and (iii) in Section 48.3). The stage bound deducts a logarithmic counting term from the predicted divisibility of the diagram of the lift and credits each erased box; we compare it with gain$(p)/m$, allowing a uniform error and the collision sum. We work it out first on a model of the places, then for the data, and finally state how the bound reaches the kernel.

50.1. The stage bound

Recall (Definitions 49.2.1 and 49.2.2) that a diagram β is read at heights below a fixed bound, the number of rows; $\beta_1(\tau)$ is its number of columns of height one at τ, $\beta_{>1}(\tau)$ the number of its other columns, $\|\beta\|_{>1}(\tau)$ the number of boxes in these, and $|\beta|$ the number of all its boxes; $\beta \smallsetminus \Phi'$ deletes the columns of height one at the places of Φ'. With $\tau_e := [-p^e]\tau$,

$$\mathcal{D}(\beta) = c_p \sum_{e \in I_p} \sum_{\tau \in \Phi} \Big[\Big(\frac{\beta_1(\tau)}{p^e} - \beta_1(\tau_e) - \beta_{>1}(\tau_e) \Big)_+ + \Big(\frac{\|\beta\|_{>1}(\tau)}{p^e} - \|\beta\|_{>1}(\tau_e) \Big)_+ - \frac{\beta_{>1}(\tau)}{p^e} \Big] - [p = d]\, \frac{1}{2} \sum_{\tau \in \Phi} \mathrm{rc}(\tau),$$

with (a notation of this chapter) $\mathrm{rc}(\tau) := \min(\beta_1(\tau), \beta_1(\tau_0) + \beta_{>1}(\tau_0)) + \beta_{>1}(\tau) + \min(\|\beta\|_{>1}(\tau), \|\beta\|_{>1}(\tau_0))$ (Definition 49.2.3). The diagram of the data (Definition 34.2.1) has

$$\beta(\tau,1) = a\, \mathbf{1}_{\Phi_a}(\tau), \qquad \beta(\tau,k) = \kappa_1(\tau) - \Big\lfloor \frac{k+t+1}{2} \Big\rfloor,$$

$\beta(\tau,h) = \Delta - (t+1)$ if $h > k$ and $(t+1) \mid h-k$, and $\beta(\tau,h) = 0$ otherwise. For $0 \le j < m$ we read it at heights below n_{j+1} and keep the letter β.

Date: November 26, 2026.

Definition 50.1.1. Let p be a prime with $\chi_K(p) \neq 1$, $j \geq 0$, $\Phi' \subseteq \Phi$, and β the diagram of the data read at heights below n_{j+1}. The *stage bound* is

$$\mathcal{D}_j(\Phi') := \mathcal{D}(\beta \smallsetminus \Phi') + \gamma_p\big(|\beta| - |\beta \smallsetminus \Phi'|\big) - f n_{j+1}^3\big(1 + \log_p \max(|\beta|, 1)\big).$$

Problem 50.1.1. Let $0 \leq j < m$. Read at heights below n_{j+1}, the diagram of the data has $\beta_1(\tau) = a\,\mathbf{1}_{\Phi_a}(\tau)$, $\beta_{>1}(\tau) \leq \kappa(\tau) + x^{60}$ and $k\kappa(\tau) \leq \|\beta\|_{>1}(\tau) \leq k\kappa(\tau) + x^{60}$ for every $\tau \in \Phi$.

50.2. A model on $(\mathbb{Z}/40)^\times$

For $\tau \in \Phi_a$ and $e \in I_p$, the first positive part in $\mathcal{D}(\beta)$ compares a/p^e with $\beta_1(\tau_e) + \beta_{>1}(\tau_e)$; so does $s_{\mathrm{fin},p}(a, \tau)$, with $\kappa(\tau_e)$ in place of the columns. We follow the comparison on a model.

Problem 50.2.1. By Problem 49.1.1, $c_p \sum_{e \in I_p} p^{-e} = \gamma_p$. Deduce that $\gamma_p = p/(p^2 - 1)$ if $\chi_K(p) = -1$, and $\gamma_p = 1/(2d - 2)$ if $p = d$.

The *model*: $\Phi = (\mathbb{Z}/40)^\times$ with $[\nu]$ the class of ν for $(\nu, 40) = 1$, $q_0 = 8$, $\Phi_a = \{\tau \equiv 1 \bmod 8\}$, $\Phi_b = \{7\}$, $x = 6$, $b = 343$, $a = xb = 2058$, $x_1 = 49$, $k = 2$, $\kappa = b$ on Φ_b and $\kappa = a/x_1 = 42$ elsewhere. The primes 3, 7 are treated as inert: $c_p = 1$, I_p is the set of odd $e \geq 1$, and $\gamma_p := c_p \sum_{e \in I_p} p^{-e} = p/(p^2 - 1)$. The *model diagram* has a columns of height one at each a-place, $\kappa(\tau)$ columns of height k at every τ, and no others; it is read at heights below 3. These numbers are not data ($18480 \nmid 8$); the model keeps only the bookkeeping.

Problem 50.2.2. In the model, -3 has order 2 modulo 8, and at $p = 3$ no pair is a collision. At $p = 7$, $-7 \equiv 33 \pmod{40}$, the powers of 33 are $\{1, 33, 9, 17\} = \Phi_a$, and every (τ, e) with $\tau \in \Phi_a$ and e odd is a collision outside $\varnothing$.

Problem 50.2.3. In the model, $\mathrm{gain}(3)/m = 25718/9$ and $\mathrm{gain}(7)/m = 1022$; moreover $C_7(\varnothing) = 2401/2$ and $C_7(\{1, 9\}) = 0$.

In the next three problems β is any diagram with $\beta_1 = a\,\mathbf{1}_{\Phi_a}$ and, for some real $\varepsilon \geq 0$ (a letter of this chapter only) and every τ, $\beta_{>1}(\tau) \leq \kappa(\tau) + \varepsilon$ and $k\kappa(\tau) \leq \|\beta\|_{>1}(\tau) \leq k\kappa(\tau) + \varepsilon$; $\Phi' \subseteq \Phi$, and β_1, $\beta_{>1}$, $\|\beta\|_{>1}$ in the terms of $\mathcal{D}(\beta \smallsetminus \Phi')$ are those of $\beta \smallsetminus \Phi'$ (only β_1 changes). The model diagram is the case $\varepsilon = 0$ with equalities.

Problem 50.2.4. $(\|\beta\|_{>1}(\tau)/p^e - \|\beta\|_{>1}(\tau_e))_+ \geq k(\kappa(\tau)/p^e - \kappa(\tau_e))_+ - \varepsilon$, with equality and $\varepsilon = 0$ for the model diagram.

Problem 50.2.5. Let $\tau \in \Phi_a \smallsetminus \Phi'$ and $e \in I_p$. If $\tau_e \notin \Phi_a \smallsetminus \Phi'$, the first positive part inside the brackets defining $\mathcal{D}(\beta \smallsetminus \Phi')$ at (τ, e) is at least $(a/p^e - \kappa(\tau_e))_+ - \varepsilon$. If (τ, e) is a collision outside Φ', it is 0, which is at least $(a/p^e - \kappa(\tau_e))_+ - a/p^e$.

Problem 50.2.6. Let $1 \in I_p$, $p < x_1$, and $\kappa \geq a/x_1$ on Φ. For every τ, $\gamma_p a \geq c_p \sum_{e \in I_p} (a/p^e - \kappa(\tau_e))_+ + c_p a/x_1$.

Problem 50.2.7. In the model, let $p \in \{3, 7\}$, β the model diagram and $\Phi' \subseteq \Phi$. Then

$$\mathcal{D}(\beta \smallsetminus \Phi') + \gamma_p a\,\#(\Phi_a \cap \Phi') \;\geq\; \frac{\mathrm{gain}(p)}{m} - C_p(\Phi') + \#(\Phi_a \cap \Phi')\,\frac{a}{x_1} - \gamma_p \sum_{\tau \in \Phi} \kappa(\tau).$$

Problem 50.2.8. In the model, at $p = 3$ and $\Phi' = \varnothing$ both sides of Problem 50.2.7 equal 179473/72. At $p = 7$ the left side is $-6139/48$ for $\Phi' = \varnothing$ (the right side is $-15379/48$); for $\Phi' = \{1, 9\}$ it is 46865/48 and the right side is 46277/48, a slack of 49/4.

50.3. The data

Now let β be the diagram of the data read at heights below n_{j+1}, $0 \le j < m$, and let $p < x_1$ be a prime with $\chi_K(p) \neq 1$. In this section take $\varepsilon = x^{60}$.

Problem 50.3.1. (i) If $p^e \ge x_1$, then $(a/p^e - \kappa(\tau_e))_+ = 0$ and $(\kappa(\tau)/p^e - \kappa(\tau_e))_+ = 0$; and $2c_p\lceil \log_p x_1 \rceil x^{60} \le a/(pm)$. (ii) $\gamma_p \beta_{>1}(\tau) \le a/(pm)$ for every τ. (iii) $n_{j+1}^3(1 + \log_p \max(|\beta|, 1)) \le a/(pm)$.

Problem 50.3.2. Let $p = d$. For every τ, the places τ and $\tau_0 = [-1]\tau$ are neither both in Φ_a nor both in Φ_b, and $\mathrm{rc}(\tau) \le 3a/(dm)$, also for $\beta \smallsetminus \Phi'$ in place of β. Hence the correction $\frac{1}{2}\sum_\tau \mathrm{rc}(\tau)$ is at most $2fa/(dm)$.

Problem 50.3.3. For every $\Phi' \subseteq \Phi$, $\mathcal{D}_j(\Phi') \ge \mathrm{gain}(p)/m - C_p(\Phi') + c_p\,\#(\Phi_a \cap \Phi')\,a/x_1 - 5fa/(pm)$. Deduce part (b) of Theorem 50.4.4.

50.4. One kernel

Fix a twist ϑ and $j \ge 0$. A *datum* of the lift is an element of the finite part of the Schwartz space of $\mathrm{Hom}(X_j, X_{j+1})$, the complex span of the finite Heisenberg translates of the vacuum. For a datum φ, let $\mathcal{J}_\varphi$ denote, in this chapter, the kernel of the definite lift from X_j to X_{j+1} built from ϑ, evaluated at φ (Section 43.4): a linear map from the Weyl vectors of the weight of X_j to those of X_{j+1}. Its diagram is β read at heights below n_{j+1}. Fix a prime p and a ring map $\eta\colon \mathbb{C} \to \mathbb{C}_p$.

Definition 50.4.1. Let $\delta \in \mathcal{O}_K$. A *standard datum of denominator* δ on $\mathrm{Hom}(X_j, X_{j+1})$ is an element of the $\mathcal{O}_\eta$-span of the translates by $\mathrm{Heis}(y)$ of the integral vacuum, over the finite adelic y such that δy is integral at every finite place of E above p.

Definition 50.4.2. A datum φ *multiplies gauges by at most* $c \in \mathbb{R}$ if $\|\mathcal{J}_{(g,g')\varphi}\,\omega\|_{g'} \le \max(c, 0)\,\|\omega\|_g$ for all finite adelic points g of the unitary group of X_j, g' of that of X_{j+1}, and all Weyl vectors ω; here $(g, g')\varphi$ is the translate of φ under the Weil representation.

Definition 50.4.3. The *jet series* of a datum φ at scale $c_0 \in \mathbb{C}$ is the Taylor series at 0 of $z \mapsto \mathrm{th}_\varphi(c_0 z)$, where th_φ is the theta function of φ, an entire function of a family $z = (z_\tau)_{\tau\in\Phi}$ of $n_{j+1} \times n_j$ matrices in the standard bases of X_{j+1} and X_j. The *jet scale* is $c_{\mathrm{jet}} := c_{\mathrm{per}}\sqrt{2\pi/\sqrt{d}}/\Omega_0$ (`Parameters.jetScale`), where c_{per} is the period factor fixed in the proof (`Parameters.periodFactor`) and Ω_0 (a letter of this chapter) is the complex number fixed in the proof (`Parameters.cmPeriod`) such that $\Omega_0\mathcal{O}_K$ is the period lattice of the CM curve of K (`Parameters.latticeModule_cmCurve`). A *jet* of a series F is $Q(\partial)F(0)$ for a polynomial Q.

The kernel of the lift is the theta kernel at scale $\pi_M c_{\mathrm{jet}}$ (`Data.liftKernel`), and $|\eta\pi_M| \le 1$ (`Data.norm_star_piMBar_le_one`) [GL]. Four statements are read from the proof [GL].

(I) Integrality. If $\delta \neq 0$ and φ is standard of denominator δ, then every jet of its jet series at scale $\bar{\delta} dc_{\mathrm{jet}}$, along a polynomial with coefficients in $\mathcal{O}_\eta$, is integral at η.

Problem 50.4.1. Let F be a power series over $\mathbb{C}_p$, $c' \in \mathbb{C}_p$, and Q homogeneous of degree l over $\mathbb{C}_p$. Then $Q(\partial)[F(c'z)](0) = c'^l Q(\partial)F(0)$. Hence, under (I), for $c \in \mathbb{C}$ and homogeneous Q of degree l with coefficients in $\mathcal{O}_\eta$, the jet along Q of the jet series of φ at scale cc_{jet} has absolute value at most $|\eta(c/(\bar{\delta}d))|^l$ after η.

(II) Kernel and jets. Let $c_0 \in \mathbb{C}$ and $\mathfrak{b} \in \mathbb{R}$. If every entry of $\mathrm{schur}_\beta(\partial)F(0)$ has absolute value at most $\mathfrak{b}$ after η, for the jet series F at scale c_0 of every standard datum of denominator δ, then every standard datum of denominator δ multiplies the gauges of the theta kernel at scale c_0 by at most $\mathfrak{b}$. Here $\mathrm{schur}_\beta(z)$ is the matrix, in the bases of standard fillings, of the map of Weyl modules of shape β induced by z; its entries are polynomials with integral coefficients (`schur`; for the change of coefficients `map_schur`).

(III) Schur entries. If no column of β has height strictly between n_j and n_{j+1} (note $n_j < n_{j+1}$), then for every series F over $\mathbb{C}_p$ and every entry of $\mathrm{schur}_\beta(\partial)F(0)$ there is an integral family μ such that the absolute value of the entry is at most $|P_\mu(\partial)F(0)|$, for the product of minors P_μ of the output filling indexing the row of that entry.

Problem 50.4.2. A polynomial over $\mathcal{O}_{\mathbb{C}_p}$ in finitely many variables with a unit coefficient takes a unit value at some point with integral coordinates.

Problem 50.4.3. For every $j \geq 0$, the diagram of the data has no column of height h with $n_j < h < n_{j+1}$.

(IV) Normal form. Let $\chi_K(p) \neq 1$, φ a standard datum of denominator 1 and $c \in \mathbb{C}$ with $|\eta c| \leq 1$. Then the jet series of φ at scale cc_{jet}, read in $\mathbb{C}_p$ through η, has the normal form of Section 17.5 for the exponents I_p°, the partner rule $([-p^e]\cdot)_e$, some ϖ with $\operatorname{val}\varpi = 1/(p-1) - \gamma_p$, and some integral family of coefficients.

Status: (I)–(IV) are read from the proof; (III) uses the standard basis theorem [DEP] and (IV) Lubin–Tate theory [LT]; for the formal group of an elliptic curve see [AEC, Ch. IV, §§1–2].

Problem 50.4.4. Let $\chi_K(p) \neq 1$. For every standard datum of denominator 1, with jet series F at scale $\pi_M c_{\mathrm{jet}}$ read through η, every standard filling and every integral family μ,

$$\operatorname{val}\big(P_\mu(\partial)F(0)\big) \;\geq\; \mathcal{D}(\beta) - f n_{j+1}^3 \big(1 + \log_p \max(|\beta|, 1)\big) = \mathcal{D}_j(\varnothing).$$

Hence every standard datum of denominator 1 multiplies gauges by at most $p^{-\mathcal{D}_j(\varnothing)}$.

This is part (a) of the following theorem; part (b) was proved in Section 50.3, and Problem 50.2.7 is its model.

Theorem 50.4.4. *Let p be a prime and $\eta\colon \mathbb{C} \to \mathbb{C}_p$ a ring map.*

(a) *If $\chi_K(p) \neq 1$, then for every twist ϑ, every $j \geq 0$ and every standard datum φ of denominator 1 on $\mathrm{Hom}(X_j, X_{j+1})$, φ multiplies gauges by at most $p^{-\mathcal{D}_j(\varnothing)}$.*

(b) *If $p < x_1$ and $\chi_K(p) \neq 1$, then for every $0 \leq j < m$ and every $\Phi' \subseteq \Phi$,*

$$\mathcal{D}_j(\Phi') \;\geq\; \frac{\mathrm{gain}(p)}{m} - C_p(\Phi') + c_p \,\#(\Phi_a \cap \Phi')\,\frac{a}{x_1} - \frac{f a (\log b)^2}{mp}.$$

Problem 50.4.4 deduces part (a) from (II)–(IV) and Theorem 17.5.1. Statement (I) and Problem 50.4.1 give the jet integrality behind input (i) of Section 48.3. Part (a) is the single-kernel estimate in input (ii) there, and part (b) is input (iii); Problem 48.3.2 uses them with $\Phi' = \varnothing$.

50.5. The size of a collision sum

How much can collisions cost? Problem 48.4.2 of Chapter 48 uses the crude bound $mC_p(\varnothing) \le j_p := \lceil 2mfa/p \rceil$.

Problem 50.5.1. In a second model let $\Phi = (\mathbb{Z}/4)^\times$, let both places be a-places, $a = 24$, and treat $p = 5$ as inert, so that I_5 is the set of odd $e \ge 1$ and $[-5^e]$ exchanges the two places for every $e \in I_5$. Then $C_5(\varnothing) = 10$; with $f = 2$ and $m = 8$, $mC_5(\varnothing) = 80 \le j_5 = 154$.

What has been established. Theorem 50.4.4. Problem 50.4.4 deduces part (a) from (II)–(IV) and Theorem 17.5.1. Part (b), proved by Problems 50.1.1–50.3.3, bounds the shortfall of the stage bound from $\mathrm{gain}(p)/m$ by $C_p(\Phi') + fa(\log b)^2/(mp) - c_p\,\#(\Phi_a \cap \Phi')\,a/x_1$; each erased long row receives the credit $\gamma_p a$ in the definition of the stage bound. Section 48.3 adds the exponents $\mathcal{D}_j$ along the chain of m lifts, and at an exceptional prime a full mark makes the collision sum outside its induced places zero (Problem 48.4.1).

In the Lean formalization. In the Lean proof [GL] (paths relative to `grh/src/`; the namespace `New.` is omitted from names), Theorem 50.4.4(a) is `Data.mem_raises_neg_divisibilityBound` (`New/UpperBound/Divisibility/ValuationAtNonSplitPrimes.lean`) and (b) is `Data.le_divisibilityBound` (`New/UpperBound/Divisibility/JetExponent.lean`), with $\mathcal{D}_j(\Phi')$ `Data.divisibilityBound`. Standard data, "multiplies gauges by at most" and the jet series are `Parameters.standardData`, `Kernel.raises` (at base points $1, 1$) and `Space.jetSeries`. Statements (I)–(IV) specialize `Parameters.norm_jet_jetSeries_le_one`, `Parameters.omega_mem_raises_of_schurJet_le`, `exists_norm_schurJet_le_norm_jet_minorProd` and `Parameters.exists_hasNormalForm_jetSeries`; in (II) the three translating points of the declaration are taken to be 1.

Hints

§50.1.

50.1.1: Below n_{j+1} the possible heights are 1, k and $k+i(t+1)$ with $1 \le i \le j$; height 1 occurs exactly at the a-places. Use $\kappa_1 = \kappa + \frac{3}{2}\Delta$, $\Delta = 2r^8$, $r \le 3x^5$, $m \le 2\sqrt{x}$.

§50.2.

50.2.1: Two geometric series.

50.2.3: At $p = 3$ only $e \in \{1, 3\}$ contribute, at $p = 7$ only $e = 1$ to $s_{\mathrm{fin},p}$; C_7 is a geometric series.

50.2.4: $(y - z)_+$ is increasing in y and decreasing in z.

50.2.5: In the first case $\beta_1(\tau_e) = 0$ after erasure.

50.2.6: Problem 50.2.1; compare the terms $e = 1$, using $p < x_1$ and $\kappa \ge a/x_1$.

50.2.7: Problems 50.2.4–50.2.6 with $\varepsilon = 0$; the last term of $\mathcal{D}$ is $-\gamma_p \sum_\tau \kappa(\tau)$ by Problem 50.2.1.

§50.3.

50.3.1: (i) $\kappa \geq a/x_1$ and $\kappa \leq b \leq a$. (ii) $\gamma_p \leq 2/p$ and Problem 50.1.1. (iii) $|\beta| \leq fa + f(kb + x^{60})$, $n_{j+1} \leq n_m$, and $a/(pm) \geq (x/m)(b/x_1)$; the scale inequalities of Chapter 33 make x/m and b/x_1 huge.

50.3.3: Use the row comparisons only for $p^e < x_1$; Problem 50.2.6 is already summed over e. Count the low columns, the correction at d and the logarithmic term separately. For the final deduction use $(\log b)^2 \geq 5$.

50.3.2: τ_0 has class $-c$ modulo q_0 if τ has class c. Bound each of the three terms of $\mathrm{rc}(\tau)$ by $a/(dm)$.

§50.4.

50.4.1: Compare monomials; the jet series at scale cc_{jet} is that at scale $\bar{\delta}dc_{\mathrm{jet}}$ with z replaced by $(c/(\bar{\delta}d))z$.

50.4.2: Reduce modulo the maximal ideal; $\bar{\mathbb{F}}_p$ is infinite.

50.4.3: $n_j = k + j(t+1)$.

50.4.4: (IV) and Theorem 17.5.1 for each standard datum; the directions in (III) may depend on the entry, so use the common bound in (II). Extend β by $\beta(\tau, 0) = 0$ when comparing $\mathcal{D}(\beta)$ with the predicted divisibility of Theorem 17.5.1.

§50.5.

50.5.1: Count the collisions for each $e \in I_5$.

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