THE COUNT: CONGRUENCES, DIRECTIONS AND FACTORIALS

Prerequisites. Chapter 17, *Pairings on a formal group of height two*: Problems 17.3.1 and 17.3.3, and Section 17.5 entire; Problem 17.4.4. Legendre's formula $\mathrm{val}(n!) = (n - s_p(n))/(p-1)$, with $s_p(n)$ the sum of the base-p digits of n. Determinants over a commutative ring; the field $\mathbb{C}_p$, its ring of integers $\mathcal{O}_{\mathbb{C}_p}$, and the valuation val with $\mathrm{val}(p) = 1$ (Chapter 34).

In Chapter 48 the kernel bound (ii) of Section 48.3 rests on a bound for the jets of a series in the normal form along products of minors, stated in Theorem 17.5.1. For the marks at exceptional primes $p < x_1$ (Section 48.4) the proof [GL] needs it in a stronger form, in which the columns of height one at some places are removed from the prediction and paid for by powers of p on the rows. This sheet proves the bound for two places and states the general count with charges. How much of the credit $\gamma_p n$ of a jet of order n, from the factorials and the rescaling by ϖ, survives the couplings between a place and its partners?

Throughout, $d \equiv 7 \pmod 8$ is a prime, $K = \mathbb{Q}(\sqrt{-d})$, p is a prime with $\chi_K(p) = (\frac{p}{d}) \neq 1$, and c_p, I_p, I_p°, $\rho(d,p)$ are as in Section 17.5: $c_p = 1/2$ if $p = d$ and 1 otherwise, I_p° is the set of $e \geq 0$ with $p = d$, or $\chi_K(p) = -1$ and e odd, $I_p = I_p^\circ \smallsetminus \{0\}$, and $\rho(d,p) = c_p/(p^2-1)$ for p inert, $c_p/(p-1)$ for $p = d$. As in Chapter 48, $\gamma_p := 1/(p-1) - \rho(d,p)$. Let $\pi_p := p$ if $p \neq d$, and $\pi_p := \eta(\sqrt{-d})$ if $p = d$, for a fixed ring map $\eta\colon \mathbb{C} \to \mathbb{C}_p$; so $\mathrm{val}\,\pi_p = c_p$. Integral series and jets $(Q(\partial)G)(0)$ are as in Section 17.5.

49.1. Two places and one row

Problem 49.1.1. $\gamma_p = c_p \sum_{e \in I_p} p^{-e}$. If $\varpi \in \mathbb{C}_p$ has $\mathrm{val}\,\varpi = \rho(d,p)$, then $\mathrm{val}(\varpi^{p+1}\pi_p/p) = 1/(p-1)$, both for p inert and for $p = d$. [Check: $\gamma_3 = 3/8$ at an inert prime 3.]

For the rest of this section $\chi_K(p) = -1$, so that $c_p = 1$, $I_p^\circ = I_p$ is the set of odd $e \geq 1$, and $\mathrm{val}\,\varpi = \rho := 1/(p^2-1)$. The *two-place model* has two places $1, 2$, one row and one input coordinate, hence two variables z_1, z_2; the partner of each place, for every e, is the other place. So the couplings are $\Gamma_e = z_1^{p^e} z_2 + z_2^{p^e} z_1$, and $F \in \mathbb{C}_p[[z_1, z_2]]$ has the normal form for (I_p, ϖ) if

$$F(\varpi z_1, \varpi z_2) = W(z_1, z_2)\, \exp\Big(\frac{\varpi^{p+1}}{p}\, \mathcal{E}(z_1, z_2)\Big),$$

with W integral and $\mathcal{E}$ a sum of the powers $\Gamma_e^{p^j}$ $(e \in I_p,\ j \geq 0)$ with coefficients in $\mathcal{O}_{\mathbb{C}_p}$. In this sheet a and w are the orders of a jet $\partial_1^a \partial_2^w$, with $\partial_i := \partial/\partial z_i$.

Problem 49.1.2. Let $p = 3$. The series $F_0 := \exp\big((z_1^3 z_2 + z_1 z_2^3)/3\big)$ has the normal form with $W = 1$ and $\mathcal{E} = \Gamma_1$. Its jets at $(a, w) = (3k, k)$ are $(3k)!/3^k$, that is $2, 80, 13440$ for $k = 1, 2, 3$, of valuations $0, 0, 1$.

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The credit comes from the factorials.

Problem 49.1.3. For integers $\nu_1, \dots, \nu_q \geq 0$ with $\nu_1 + \cdots + \nu_q = n$, and $\nu! := \nu_1! \cdots \nu_q!$,

$$\operatorname{val}(\nu!) \;\geq\; \frac{n}{p-1} - q\big(1 + \lfloor \log_p n \rfloor\big),$$

with $\lfloor \log_p 0 \rfloor := 0$. [Check: $p = 3$, $\nu = (27, 1)$: $13 \geq 14 - 8$.]

Problem 49.1.4. Let G be a power series over $\mathbb{C}_p$ in q variables whose coefficients of degree n are integral, and let Q be a homogeneous polynomial of degree n with coefficients in $\mathcal{O}_{\mathbb{C}_p}$. Then $\operatorname{val}\big((Q(\partial)G)(0)\big)$ is at least the bound of Problem 49.1.3.

The debit comes from the couplings. One differentiates along scaled directions.

Problem 49.1.5. For $y_1, y_2 \in \mathbb{C}_p$ and $G(u_1, u_2) := F(\varpi y_1 u_1, \varpi y_2 u_2)$, $(\partial_{u_1}^a \partial_{u_2}^w G)(0) = \varpi^{a+w} y_1^a y_2^w \, (\partial_1^a \partial_2^w F)(0)$.

Problem 49.1.6. Let F have the normal form, $n \geq 0$, and $y_1, y_2 \in \mathcal{O}_{\mathbb{C}_p}$ with $y_1^{p^e} y_2 \equiv y_2^{p^e} y_1 \equiv 0 \pmod{p}$ for every $e \in I_p$ with $p^e \leq n$. Then the coefficients of degree at most n of $F(\varpi y_1 u_1, \varpi y_2 u_2)$ are integral.

Problem 49.1.7. Fix $e \in I_p$, and let $p^{1/p^e} \in \mathcal{O}_{\mathbb{C}_p}$ be a root of $X^{p^e} = p$. The pairs $(y_1, y_2) = (1, p)$, $(p, 1)$, $(p^{1/p^e}, p^{1/p^e})$ satisfy both congruences of Problem 49.1.6 at e, at the costs $a \operatorname{val} y_1 + w \operatorname{val} y_2 = w$, a, $(a + w)/p^e$. There is also an integral pair satisfying both congruences at the cost

$$\min(a/p^e, w) + \min(w/p^e, a).$$

[Check: $p = 3$, $(a, w) = (27, 1)$: $4/3$ at $e = 1$, $28/27$ at $e = 3$.]

Problem 49.1.8. If F has the normal form, then for all $a, w \geq 0$,

$$\operatorname{val}\big((\partial_1^a \partial_2^w F)(0)\big) \;\geq\; \gamma_p(a + w) - \sum_{e \in I_p,\ p^e \leq a+w} \Big[\min\Big(\frac{a}{p^e}, w\Big) + \min\Big(\frac{w}{p^e}, a\Big)\Big] - 2\big(1 + \lfloor \log_p(a + w) \rfloor\big).$$

Problem 49.1.9. For $a + w \geq 1$, the right side of Problem 49.1.8 is at least $\mathcal{D}(a, w) - 2(1 + \log_p(a + w))$, where

$$\mathcal{D}(a, w) := \sum_{e \in I_p} \Big[\Big(\frac{a}{p^e} - w\Big)_+ + \Big(\frac{w}{p^e} - a\Big)_+\Big]$$

is the predicted divisibility of Problem 17.5.3. [Check: $p = 3$: $\mathcal{D}(27, 1) = 8$, $\mathcal{D}(54, 2) = 16$, $\mathcal{D}(243, 1) = 88$.]

Problem 49.1.10. Let $p = 3$ and $F_1 := F_0/(1 - z_1)$. Then F_1 has the normal form, and $(\partial_1^a \partial_2 F_1)(0) = a!/3$ for $a \geq 3$, of valuation 12 at $a = 27$ and 120 at $a = 243$. Problem 17.4.4 gives valuations 11 at $(27, 1)$ and 22 at $(54, 2)$ for $\exp(-Q_{c,\sigma c})$, c a unit. All four exceed $\mathcal{D}(a, w)$.

Problems 49.1.3–49.1.9 prove the count for two places, one row and one input coordinate, with the error $2(1 + \log_p \max(a + w, 1))$. The model is not an instance of Theorem 17.5.1: there the partners are the places $[-p^e]\tau$, and only columns of height less than the number n of rows count, so height-one columns need $n \geq 2$; with $n = 2$ and all boxes in row 0 the error is $8f(1 + \log_p \max(a + w, 1))$. In general a direction has one component per input coordinate, and at height h the proof takes integral combinations of the first h directions by an $h \times h$ matrix.

49.2. Diagrams, directions and the cost

From now on we are in the setting of Theorem 17.5.1: q_0, d, s, the set Φ of infinite places of $\mathbb{Q}(\sqrt{-d},\zeta_{q_1})$ and $f=\#\Phi$ are as there, p is a prime with $\chi_K(p)\neq 1$, and the partner of τ at e is $[-p^e]\tau$. There are n rows $0\le i<n$ and n' input coordinates $0\le l<n'$, and the variables are $z_{\tau il}$. The couplings are $\Gamma_e=\sum_{\tau,i,l} z_{\tau il}^{p^e} z_{[-p^e]\tau,il}$, and $\varpi\in\mathbb{C}_p$ has $\operatorname{val}\varpi=\rho(d,p)$. We say that $F\in\mathbb{C}_p[[z]]$ has the normal form if it has the normal form of Section 17.5 for $(I_p^\circ,[-p^e]\cdot\,,\varpi)$ and some family of integral coefficients.

Definition 49.2.1. A *diagram* is a function $\beta\colon \Phi\times\{1,2,\dots\}\to\mathbb{N}$; $\beta(\tau,h)$ is the number of columns of height h at τ. Only the heights $h<n$ count:

$$\beta_1(\tau):=\beta(\tau,1),\qquad \beta_{>1}(\tau):=\sum_{1<h<n}\beta(\tau,h),\qquad \|\beta\|_{>1}(\tau):=\sum_{1<h<n}h\,\beta(\tau,h),$$

(with $\beta_1:=0$ if $n\le 1$), and the number of boxes is $|\beta|:=\sum_\tau\big(\beta_1(\tau)+\|\beta\|_{>1}(\tau)\big)$.

Definition 49.2.2. For $\Phi'\subseteq\Phi$, the *erasure* $\beta\smallsetminus\Phi'$ is the diagram equal to 0 at $(\tau,1)$ for $\tau\in\Phi'$ and to β elsewhere.

Definition 49.2.3. With $\tau':=[-p^e]\tau$ and $\tau_0:=[-1]\tau$, the *predicted divisibility* of β is

$$\begin{aligned}\mathcal{D}(\beta):=c_p\sum_{e\in I_p}\sum_{\tau\in\Phi}\Big[&\Big(\frac{\beta_1(\tau)}{p^e}-\beta_1(\tau')-\beta_{>1}(\tau')\Big)_+\\ &+\Big(\frac{\|\beta\|_{>1}(\tau)}{p^e}-\|\beta\|_{>1}(\tau')\Big)_+-\frac{\beta_{>1}(\tau)}{p^e}\Big]\\ -[p=d]\,\frac12\sum_{\tau\in\Phi}\Big[&\min\big(\beta_1(\tau),\beta_1(\tau_0)+\beta_{>1}(\tau_0)\big)+\beta_{>1}(\tau)\\ &+\min\big(\|\beta\|_{>1}(\tau),\|\beta\|_{>1}(\tau_0)\big)\Big],\end{aligned}$$

where $[p=d]$ is 1 if $p=d$ and 0 otherwise.

For the shape κ with $\kappa(\tau,0)=0$ and $\kappa=\beta$ at heights $h\ge 1$ this is the number $\Delta(\kappa)$ of Section 17.5, and a_τ,a'_τ,l_τ there are $\beta_1(\tau),\beta_{>1}(\tau),\|\beta\|_{>1}(\tau)$. For a standard filling $\mathbf{i}$, which puts the row $\mathbf{i}(\tau,h,c,j)$ into box $j<h$ of the cth column $(1\le c\le\beta(\tau,h))$ of height h at τ, $1\le h<n$, and integral directions $y=(y_{\tau lk})$ $(k\ge 0)$, the product of minors of Section 17.5, written P_μ there for the family $\mu=y$, is

$$P_y(z)=\prod_{\tau\in\Phi}\prod_{h<n}\prod_{1\le c\le\beta(\tau,h)}\det\Big(\sum_l y_{\tau lk}\,z_{\tau,\mathbf{i}(\tau,h,c,j),l}\Big)_{0\le j,k<h}.$$

Three facts of the general step are read from the proof (see the remark). The proof replaces the given directions by new ones, $y'_{\tau hk}:=\sum_{k'<h}(g_{\tau h})_{kk'}\,y_{\tau\cdot k'}$, with one integral $h\times h$ matrix $g_{\tau h}$ for each place and height, shared by the $\beta(\tau,h)$ columns.

Problem 49.2.1. Let $v_1,\dots,v_h\in\mathcal{O}_{\mathbb{C}_p}$ be the values of a linear form on the h rows of a matrix, and $\varpi'\in\mathcal{O}_{\mathbb{C}_p}$. Some $g\in M_h(\mathcal{O}_{\mathbb{C}_p})$ with $\det g\mid\varpi'$ has $\varpi'\mid(gv)_k$ for every k. For finitely many forms and moduli there is one g, with $\det g$ dividing the product of the moduli, that works for all of them. [Over $\mathbb{Z}_3$, for $v=(1,2)^t$ and $\varpi'=3$, there is such a g with $gv=(3,0)^t$ and $\det g=3$.]

Problem 49.2.2. Take one place (dropped from indices), $n = 3$, $n' = 2$, two columns of height 2 filled with the rows 0, 1, and the coordinate vectors as directions. Then $P_y = (z_{00}z_{11} - z_{01}z_{10})^2$ and $(P_y(\partial)P_y)(0) = 12$. With the first direction scaled by 3, $y = \mathrm{diag}(3, 1)$, the jet of $(z_{00}z_{11} - z_{01}z_{10})^2$ along P_y is 108.

In general put $z_{\tau il} = \varpi \sum_{h<n,\ k<h} u_{\tau hki}\, y'_{\tau hk,l}$ in new variables $u_{\tau hki}$, and let Q be the product, over the columns, of $\det(u_{\tau hk\,\mathbf{i}(\tau,h,c,j)})_{0\le j,k<h}$. First, by multilinearity of the determinants and the chain rule,

$$\big(Q(\partial_u)\, F(z(u))\big)(0) = \varpi^{|\beta|} \prod_{\tau\in\Phi} \prod_{h<n} (\det g_{\tau h})^{\beta(\tau,h)}\ \big(P_y(\partial)F\big)(0),$$

the case of Problem 49.1.5 with matrices in place of scalars. Second, for a shift e, a place τ and its partner $\tau' = [-p^e]\tau$, let

$$\mathrm{ded}_{e,\tau}(\beta) := \min\Big(\frac{\beta_1(\tau)}{p^e},\ \beta_1(\tau') + \beta_{>1}(\tau')\Big) + \frac{\beta_{>1}(\tau)}{p^e} + \min\Big(\frac{\|\beta\|_{>1}(\tau)}{p^e},\ \|\beta\|_{>1}(\tau')\Big).$$

The proof chooses the $g_{\tau h}$ so that π_p divides $\sum_l y'^{\,p^e}_{\tau hk,l}\, y'_{\tau' h'k',l}$ for all $e \in I_p^\circ$ with $e \le \lfloor \log_p|\beta| \rfloor$, all places, heights and directions, and so that

$$\sum_{\tau\in\Phi} \sum_{h<n} \beta(\tau,h)\, \mathrm{val}(\det g_{\tau h}) \ \le\ c_p \sum_{e\in I_p^\circ,\ e\le\lfloor \log_p|\beta|\rfloor} \ \sum_{\tau\in\Phi} \mathrm{ded}_{e,\tau}(\beta).$$

At the source the modulus is π_p^{1/p^e}, at the partner π_p, as in Problem 49.1.7; a source condition $\sum_l y_l^{p^e} v_l \equiv 0 \pmod{\pi_p}$ follows from the linear condition $\sum_l v_l^{1/p^e} y_l \equiv 0 \pmod{\pi_p^{1/p^e}}$, so Problem 49.2.1 applies. Each height is weighted by its number of columns: the first minimum compares entry conditions at height one on the source with pairing conditions at the partner, the middle term counts source pairing conditions at the higher heights, and the last minimum compares entry conditions at those heights on the two sides. Third, π_p divides $\Gamma_e(z(u)/\varpi)$ for these e, the series $F(z(u))$ is integral through degree $|\beta|$, and Q is homogeneous of degree $|\beta|$.

Problem 49.2.3. $c_p \sum_{e\in I_p^\circ} \sum_{\tau\in\Phi} \mathrm{ded}_{e,\tau}(\beta) = \gamma_p|\beta| - \mathcal{D}(\beta)$, the series converging. The variables $u_{\tau hki}$ number at most fn^3.

As $\mathrm{ded}_{e,\tau} \ge 0$, the truncated sum is at most the full one. So Problems 49.1.3, 49.1.4 and 49.2.3, with the three facts read from the proof, give Theorem 17.5.1 for diagrams (with $\lfloor \log_p 0 \rfloor := 0$).

Problem 49.2.4. Let $p = d$ and $n \ge 2$, let σ_1 and $\sigma_2 = [-1]\sigma_1$ be two places, and let $\beta(\sigma_1, 1) = 12$, $\beta(\sigma_2, 1) = 4$ and $\beta = 0$ otherwise. The shift $e = 0$ debits $c_p\big(\mathrm{ded}_{0,\sigma_1}(\beta) + \mathrm{ded}_{0,\sigma_2}(\beta)\big) = 4$, and the half sum in $\mathcal{D}(\beta)$ contributes -4.

49.3. Charges

At an exceptional prime $p < x_1$ the loss allowed in Section 48.2 need not cover the collisions between a-places. The proof then removes the columns of height one at a set Φ' of places from the prediction and pays for their pairings with factors p on rows.

Definition 49.3.1. Let β be a diagram and $\Phi' \subseteq \Phi$. *Charges* for (β, Φ') are numbers $\mathfrak{c}(\tau, h, i) \in \mathbb{N}$ ($\tau \in \Phi$, $h \ge 1$, $0 \le i < n$) such that, for all $e \in I_p^\circ$, $\tau \in \Phi$, $h \ge 1$ and rows i, with $\tau' = [-p^e]\tau$: if both $\beta(\tau, 1) \ne 0$ and $\beta(\tau', h) \ne 0$, and if

either $\tau \in \Phi'$ or both $h = 1$ and $\tau' \in \Phi'$, then $\mathfrak{c}(\tau,1,i) + \mathfrak{c}(\tau',h,i) > 0$. Here h runs over all heights, including $h \ge n$. For a filling $\mathbf{i}$, the sum of the charges over the boxes is $\sum_{\text{boxes}} \mathfrak{c} := \sum_\tau \sum_{h<n} \sum_{1\le c\le\beta(\tau,h)} \sum_{j<h} \mathfrak{c}\big(\tau,h,\mathbf{i}(\tau,h,c,j)\big)$.

Problem 49.3.1. In the two-place model let F have the normal form. (i) $\operatorname{val}\big((\partial_1^a\partial_2^w F)(0)\big) \ge \gamma_p(a+w) - w - 2(1+\lfloor \log_p(a+w)\rfloor)$. (ii) With a second, empty row ($n = 2$), $\beta(1,1) = a$, $\beta(2,1) = w$, $\beta = 0$ otherwise, $\Phi' = \{1\}$, $\mathfrak{c}(2,1,i) = 1$ for both rows i and $\mathfrak{c} = 0$ otherwise, the numbers $\mathfrak{c}$ are charges for (β,Φ') for the partner rule of the model, and for the filling with all boxes in row 0 the main term $\mathcal{D}(\beta \smallsetminus \Phi') + \gamma_p(|\beta| - |\beta \smallsetminus \Phi'|) - \sum_{\text{boxes}} \mathfrak{c}$ equals $\gamma_p(a+w) - w$. [Check the main term in (ii) at $p = 3$: 719/8 at $(243,2)$, against $\mathcal{D}(243,2) = 86$; 2177/8 at $(729,2)$, against $\mathcal{D}(729,2) = 267$.]

In general the substitution becomes $z_{\tau il} = \varpi \sum_{h,k} p^{\mathfrak{c}(\tau,h,i)}\, u_{\tau hki}\, y'_{\tau hk,l}$, and the multiplier gains the factor $p^{\sum_{\text{boxes}} \mathfrak{c}}$. The pairings of height one at τ with height h at τ', where $\tau \in \Phi'$ or both $h = 1$ and $\tau' \in \Phi'$, are *exempt*: no congruence is imposed on them, and the charge condition gives a factor p on one side of each row (after setting the charges to 1 where $\beta(\tau,h) = 0$, which changes no box). With S the set of $e \in I_p^\circ$, $e \le \lfloor \log_p|\beta|\rfloor$, the remaining congruences give $\sum_\tau \sum_{h<n} \beta(\tau,h) \operatorname{val}(\det g_{\tau h}) \le c_p \sum_{e\in S} \sum_\tau \operatorname{ded}_{e,\tau}(\beta \smallsetminus \Phi')$, the cut-off using $|\beta|$.

Problem 49.3.2. (m) Let $q_0 = 18480$, $d = 138599$, $s = 0$, $p = 13$. By machine: d is prime, $q_0 \mid 2(d+1)$, $(d+1) \bmod q_0 = 9240$, $\chi_K(13) = -1$, $f = \varphi(4490640) = 933120$, and $[-13^e] \ne 1$ for odd e. Take $n = 2$ rows, one input coordinate, $A := 13^{12}$ columns of height one at the identity place, all in row 0, the direction 1 at that place, and $F := (z/\varpi)^A$ in the coordinate z of that row. Then F has the normal form, $\operatorname{val}\big(F^{(A)}(0)\big) = (A-1)/12 - A/168 = 302875106592239/168$, and the bound of Theorem 17.5.1 is $13A/168 - 933120 \cdot 2^3 \cdot 13 = 302858803119613/168$.

We take the following theorem from the proof. The construction of the directions uses the simultaneous elimination of Problem 49.2.1; the proof uses both conclusions of Problem 49.2.3, applying the identity to the erased diagram; Problem 49.3.1 treats one marking in the two-place model.

Theorem 49.3.2. *Let q_0, d, s, Φ, f, p with $\chi_K(p) \ne 1$, η, the n rows and the n' input coordinates be as above, let $\varpi \in \mathbb{C}_p$ with $\operatorname{val}\varpi = 1/(p-1) - \gamma_p$, and let $F \in \mathbb{C}_p[[z]]$ have the normal form. Let β be a diagram, $\Phi' \subseteq \Phi$, and $\mathfrak{c}$ charges for (β,Φ'). Then for every standard filling $\mathbf{i}$ of β and all integral directions y,*

$$\operatorname{val}\big((P_y(\partial)F)(0)\big) + \sum_{boxes} \mathfrak{c} \;\ge\; \mathcal{D}(\beta\smallsetminus\Phi') + \gamma_p\big(|\beta| - |\beta\smallsetminus\Phi'|\big) - fn^3\big(1+\log_p \max(|\beta|,1)\big).$$

For $\Phi' = \varnothing$ and $\mathfrak{c} = 0$ it gives Theorem 17.5.1 for shapes with $\kappa(\tau,0) = 0$; that theorem keeps its place in Chapter 17. The theorem is read from the proof; which series have the normal form is a separate statement.

In the Lean formalization. In the Lean proof [GL], Theorem 49.3.2 is `New.Parameters.norm_jet_minorProd_le_of_charges` in `grh/src/New/Derivatives/Divisibility/Count.lean`, stated as an upper bound for the norm of the jet, for diagrams κ on $\Phi \times \mathbb{N}$; this sheet takes $\kappa(\tau,0) = 0$ (when $n > 0$, columns of height 0 would count among the low columns). In the same file, erasure, the predicted divisibility and the sum of the charges over the boxes are `eraseHeightOne`, `predictedDivisibility` and

`fillingCharge`; the multiplier is `jet_columnDetProd_subst`; the divisibility of the substituted couplings, exempt pairs included, is `dvd_subst_coupling_directionSubst_of_charges`; and the identity of Problem 49.2.3 is `mul_tsum_deductionAt`. In `NullFrames.lean`, $\mathrm{ded}_{e,\tau}$ is `deductionTerm` with no marked places, and the change of directions with its determinant bound is `exists_newDirection_dvd`. The integrality through degree $|\beta|$ is `HasNormalForm.norm_coeff_subst_le_one` in `NormalForm.lean`.

This sheet has proved the two-place bounds, without marking and with place 1 marked (Problems 49.1.9 and 49.3.1); Theorem 49.3.2 is read from the proof. Chapter 50 uses the normal form of the jet series of a standard datum and the uncharged count to bound the kernel of one lift; the charged count enters the estimate at a mark of an exceptional prime $p < x_1$ [GL].

Hints

§49.1.

49.1.1: A geometric series; then $\mathrm{val}(\varpi^{p+1}) + c_p - 1$.
49.1.2: Only $(z_1^3 z_2)^k$ contributes to the monomial $z_1^{3k} z_2^k$.
49.1.3: Legendre's formula; $s_p(\nu_i) \le (p-1)(1 + \lfloor \log_p n \rfloor)$.
49.1.4: The jet of G along u^ν is $\nu!\,[u^\nu]G$.
49.1.6: Couplings with $p^e > n$ have degree $> n$. In degrees $\le n$ the exponent has coefficients of valuation $\ge 1/(p-1)$ (Problem 49.1.1); argue as in Problem 17.3.3.
49.1.7: For each congruence compare a source scaling by p^{1/p^e} with a partner scaling by p, and multiply the two choices.
49.1.8: Take the product of the pairs of Problem 49.1.7 over $e \in I_p$, $p^e \le a + w$; then Problems 49.1.5, 49.1.6 and 49.1.4, and $\gamma_p + \rho = 1/(p-1)$.
49.1.9: Compare $z - \min(z, z')$ with $(z - z')_+$, and expand $\gamma_p(a + w)$ by Problem 49.1.1.
49.1.10: Rescale the denominator of F_1. For the mixed jets, compare the valuations of Problem 17.4.4 directly with $\mathcal{D}(a, w)$.

§49.2.

49.2.1: Treat $h = 0$ separately; otherwise look at an entry of least valuation. For several forms, take the next form on the current rows.
49.2.3: Problem 49.1.9 for each term, $\sum_\tau (\beta_1(\tau) + \|\beta\|_{>1}(\tau)) = |\beta|$, and Problem 49.1.1; treat $e = 0$ separately. For the variables, $\sum_{h<n} h \le n^2$.

§49.3.

49.3.1: (i) Use $y_1 = 1$, $y_2 = p$ in Problems 49.1.5, 49.1.6 and 49.1.4. (ii) $\mathcal{D}(\beta \smallsetminus \Phi')$ has only the terms $(w/p^e)_+$.
49.3.2: Sage. No odd shift returns to the identity place, so $\mathcal{D} = \gamma_{13} A$.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.