THEOREM C

Prerequisites. Hadamard's and Fischer's inequalities for positive definite Hermitian matrices [HJ, §7.8]. From Chapter 5 heights, covolume and Problem 5.3.2; from Chapter 6 the theorem on a twist for each prime; from Chapter 33 the data and Problem 33.4.6; from Chapter 34 the objects and the target lattice; and the statements of the auxiliary forms, the factorization of the chain determinant and the bound for small ratios (Chapter 55), of small ratios and the cost of one lift (Chapter 60), and of regular families (Chapter 67). Section 54.1 uses no earlier chapter.

Theorem C gives the lower bound used in Chapter 35, *The main inequality*, for the covolume plus twice the index of a linearly independent family of target forms spanning almost all of the target. For each prime ℓ and each field isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, the proof chooses a twist θ_ι. It compares the chain Gram determinant at the trivial twist with its value at θ_ι, using a height bound for the family of ratios for each single lift. We first compare index and Gram determinant in a lattice, then state and prove Theorem C in the form used in the proof. This chapter heads Part IX; the chapters that follow it develop its inputs.

54.1. The comparison in a lattice

Let $N \geq 1$ be an integer, and give $\mathbb{C}^d$ the Hermitian form $\langle v, w\rangle := N\,\bar{v}^{\mathsf{T}} w$. For $y_1, \dots, y_n \in \mathbb{Q}^d$ write $\mathrm{Gram}(y) := \det(\langle y_i, y_j\rangle)_{i,j\leq n}$, and let $H := \log N$.

Problem 54.1.1. If $y_1, \dots, y_d \in \mathbb{Z}^d$ are linearly independent, then $\mathrm{Gram}(y) \geq N^d$, so $\log \mathrm{Gram}(y) \geq dH$. For $N = 5$, $y_1 = (1, 2)$, $y_2 = (2, -1)$ one has $\mathrm{Gram}(y) = 625$.

Problem 54.1.2. Let $y_1, \dots, y_d \in \mathbb{Q}^d$ be linearly independent, and let $u \in M_d(\mathbb{Z})$ with $\det u \neq 0$ be such that $\sum_k u_{ik} y_k \in \mathbb{Z}^d$ for every i. If $\log \mathrm{Gram}(y) \leq c$ and $\log|\det u| \leq \delta$, then $c + 2\delta \geq dH$. For $N = 5$, $y_1 = (\frac{1}{2}, 1)$, $y_2 = (0, \frac{1}{3})$, $u = \mathrm{diag}(2, 3)$, $c = \log \mathrm{Gram}(y)$ and $\delta = \log 6$ there is equality.

Now let the family be shorter than the dimension. Suppose it is completed to a basis by vectors of bounded length.

Problem 54.1.3. Let $y_1, \dots, y_n$ and $z_1, \dots, z_{n'}$ in $\mathbb{Z}^d$ together form a basis of $\mathbb{Q}^d$, with $\langle z_i, z_i\rangle \leq e^\beta$ for every i. Then $\log \mathrm{Gram}(y) + n'\beta \geq dH$. For $d = 3$, $N = 2$, $y_1 = (1, 1, 0)$, $y_2 = (0, 1, 1)$, $z_1 = (1, 0, 0)$: $\mathrm{Gram}(y) = 12$, $\mathrm{Gram}(y, z) = 8$, and $8 \leq 12 \cdot 2$.

Problem 54.1.4. Every linearly independent family $y_1, \dots, y_n \in \mathbb{Z}^d$ is completed to a basis of $\mathbb{Q}^d$ by $d - n$ of the standard basis vectors. Hence $\log \mathrm{Gram}(y) \geq nH$.

In Section 54.2 the main amount enters through the height of the family $\iota \mapsto Z(1)/Z(\theta_\iota)$, where the twist θ_ι is chosen for each pair (ℓ, ι) (Problem 54.2.4).

Date: November 26, 2026.

54.2. Theorem C

We use the parameters q_0, d and the data of Chapter 33, and the objects and notation of Chapter 34: the target forms T, of dimension d_T; the set Tw of m-fold twists ϑ, with the trivial twist 1 and the norm $\operatorname{Nm} Q_\vartheta$ of the modulus; the lower chain Θ_ϑ; the Gram determinants $Z(\vartheta)$ of the chain and $Z_j(\vartheta)$ of the jth single lift at ϑ^j; and families of twists $\theta = (\theta_\iota)$, one twist for each $\iota \mid \ell$. Among the conditions on the data, the scales $x < x_1 < b$ are integers with

$$(\log b)^{100} \le x, \qquad x^{10^6} \le x_1, \qquad x_1^{10^6} \le b, \qquad (\log x_1)^6 \le \log b, \qquad \log\log b \ge 10^{12},$$

the last inequality following from $10^{12}C \le \log\log b$ and $C \ge 1$, with C as in Chapter 33, Section 33.3. Further $a = xb$, $\sqrt{x} \le m \le 2\sqrt{x}$, $x^5 \le r \le 3x^5$, $k = r + 1 - m$, $t = 5r$ and $f \ge 1$. With $U = fa\log^3 b$ write $F := U \log b$. The main amount is $h_{\mathrm{main}} = mk\, h_{\mathrm{dir}}$, and $h_{\mathrm{main}} \ge 0$ by Problem 33.4.6. Heights of families $\iota \mapsto u_\iota$ are those of Chapter 5, Section 5.3; we write $\mathrm{ht}(u) \le c$.

The target lattice. The target lattice on T, with twists Tw, the lifts Θ_ϑ, injective by Problem 59.4.3, the twists allowed at ℓ, the lattices $\mathfrak{L}_{\vartheta,\iota}$, the squared lengths $\|v\|_\sigma^2$ of the σ-conjugated coordinate vectors of $\Theta_1 v$, and index and covolume of a family in T, is that of Definition 34.11.1; its volume at ϑ is $Z(\vartheta)$.

Theorem 6.2.1, for this lattice, reads as follows. Let $y_1, \dots, y_n, z_1, \dots, z_{n'}$ be a basis of T, where y has covolume at most c and index at most δ; let θ be a family of twists with θ_ι allowed at ℓ, and $\gamma\colon \mathrm{Tw} \to \mathbb{C}^\times$; suppose (i) $\gamma(\vartheta)\Theta_\vartheta z_i \in \mathfrak{L}_{\vartheta,\iota}$ for every i, every $\iota \mid \ell$ and every ϑ allowed at ℓ; (ii) $\|z_i\|_\sigma^2 \le e^{c_{\mathrm{short}}}$ for every i and σ; (iii) $\mathrm{ht}(|\gamma(\theta_\iota)|^{-2}) \le c_{\mathrm{den}}$; (iv) $\mathrm{ht}(Z(1)/Z(\theta_\iota)) \le c_{\mathrm{vol}}$. Then

$$0 \le c + 2\delta + n'(c_{\mathrm{short}} + c_{\mathrm{den}}) + c_{\mathrm{vol}}. \tag{1}$$

Four inputs feed it. (a) *Auxiliary forms* (Theorem 55.5.1). There are an integer e_γ and $\gamma\colon \mathrm{Tw} \to \mathbb{C}^\times$ with $\gamma(1) = 1$ and $|\gamma(\vartheta)| = (\operatorname{Nm} Q_\vartheta)^{e_\gamma/2} \prod_{0 \le i,j < m} \rho'_{ij}(\vartheta)$ such that T is spanned by forms z with $\gamma(\vartheta)\Theta_\vartheta z$ integral at ι whenever $\gcd(\ell, \operatorname{Nm} Q_\vartheta) = 1$ and with $\|z\|_\sigma^2 \le \exp(2x_1^2 b^2)$ for every σ. (b) *Single lifts.* There is a family θ with small ratios (Definition 64.2.2) such that $\mathrm{ht}(Z_j(1)/Z_j(\theta_\iota)) \le d_T(F/m - kh_{\mathrm{dir}})$ for $1 \le j \le m$; this combines the cost of one lift (Chapter 60) with a regular family (Chapter 67). Each θ_ι of such a family is allowed at ℓ. (c) *Factorization* (Theorem 55.4.1). There is an integer e_Z such that for every $\vartheta \in \mathrm{Tw}$

$$\frac{Z(\vartheta)}{Z(1)} = \Big((\operatorname{Nm} Q_\vartheta)^{e_Z/2} \prod_{1 \le i < j \le m} \rho_{ij}(\vartheta)\Big)^{d_T} \prod_{j=1}^{m} \frac{Z_j(\vartheta)}{Z_j(1)}.$$

(d) *Small ratios* (Theorem 55.5.2). If θ has small ratios, then for every integer e both families $\big((\operatorname{Nm} Q_{\theta_\iota})^{e/2} \prod_{0 \le i,j < m} \rho'_{ij}(\theta_\iota)\big)^{-1}$ and $\big((\operatorname{Nm} Q_{\theta_\iota})^{e/2} \prod_{1 \le i < j \le m} \rho_{ij}(\theta_\iota)\big)^{-1}$ have height at most $m^2 t x_1$.

Problem 54.2.1. The data satisfy $2m^2 t x_1 \le b$, $4(2x_1^2 + b^{-1}) \le b$, $b \le F$ and $4/b^2 \le 1/2$.

Problem 54.2.2. If $n \ge (1 - 4/b^2)d_T$ and $d_T = n + n'$, then $n'(2x_1^2 b^2 + b) \le d_T b$.

So the allowance for one completing auxiliary form is $2x_1^2 b^2 + b$, and the total allowance for the n' completing forms is at most $d_T b$.

Problem 54.2.3. For γ as in (a) and θ as in (b), $\mathrm{ht}(|\gamma(\theta_\iota)|^{-2}) \le b$.

Problem 54.2.4. For θ as in (b), $\mathrm{ht}(Z(1)/Z(\theta_\iota)) \le d_T(b + F - h_{\mathrm{main}})$.

Problem 54.2.5. Let $y_1, \dots, y_n \in T$ be linearly independent, with $n \geq (1 - 4/b^2)d_T$, covolume at most c and index at most δ. Then T has a basis $y_1, \dots, y_n, z_1, \dots, z_{n'}$ with each z_i as in (a), and $c + 2\delta \geq d_T(h_{\text{main}} - 3F)$.

Problems 54.2.1–54.2.5 prove the following.

Theorem 54.2.1 (Theorem C)**.** *Let $y_1, \dots, y_n$ be linearly independent target forms with $(1 - 4/b^2)d_T \leq n$, of index at most δ and covolume at most c. Then*

$$d_T(h_{\text{main}} - 3fa \log^4 b) \leq c + 2\delta.$$

In the Lean formalization. In the Lean proof [GL], Theorem 54.2.1 is `New.Data.integral_forms_long`, in `grh/src/New/Determinant/Twist/LongForms.lean`, and the input (b) is `New.Data.exists_hasSmallRatios_heightLe_liftDeterminant_div` in the same file; (1) is the case of the target lattice of `New.TwistedLattice.covolume_add_index_nonneg`, in `grh/src/New/Reduction/Lattices/TwistedLattice.lean`. In `LongForms.lean` the proof of (b) uses `New.exists_isRegularFamily` and `New.Data.heightLe_liftDeterminant_ratio`, and Problems 54.2.2 and 54.2.4 are `New.Data.mul_completion_cost_le` and `New.Data.heightLe_chainDeterminant_one_div`. The bounds of Problem 54.2.1 are instances of `New.Data.le_of_hasLogBound`. Problems 54.2.3 and 54.2.5 are steps in the proof of `New.Data.integral_forms_long`, the basis of Problem 54.2.5 coming from `New.exists_extend_basis_of_span_eq_top`. Theorem 54.2.1 is applied in the proof of `New.star_inequality`.

Hints

§54.1.

54.1.1: $(\langle y_i, y_j \rangle) = N Y Y^{\mathsf{T}}$ for the matrix Y with rows y_i, and $\det Y$ is a nonzero integer.

54.1.2: Problem 54.1.1 for the rows of uY.

54.1.3: Fischer's inequality for the Gram matrix of (y, z), Hadamard's for $\text{Gram}(z)$, then Problem 54.1.1.

54.1.4: Steinitz exchange and Problem 54.1.3; each standard basis vector has squared length N.

§54.2.

54.2.1: $m^2 \leq 4x$, $t \leq 15x^5$ and $x \geq 2$.

54.2.2: $n' \leq 4d_T/b^2$.

54.2.3: Input (d), Problem 5.3.2 and Problem 54.2.1.

54.2.4: Invert (c); bound the scalar by (d) and the single lifts by (b), and add by Problem 5.3.2.

54.2.5: Apply (1) with $c_{\text{short}} = 2x_1^2 b^2$, $c_{\text{den}} = b$ and $c_{\text{vol}} = d_T(b + F - h_{\text{main}})$.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[HJ] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge Univ. Press, 2013.