SPACES, FORMS AND TWISTS

Prerequisites. Chapter 33, *The data*: the parameters, the field E and its places, the scales, ranks and signs, and the level primes. From Chapter 5 the isomorphisms $\iota \mid \ell$ and the rings $\mathcal{O}_\iota$; from Chapter 6 twisted lattices (Section 6.1), their index and covolume (Section 6.2). Finite groups GL_r over finite fields and their irreducible characters, at the level of Green's parametrization; unitary groups over CM fields and their adelic points; theta lifts, at the level of definitions.

Which spaces, forms and characters does the proof attach to the data? This chapter fixes them, following [GL]; the later chapters use this notation. The Poincaré map is defined in Chapter 36, the composite lift and the data of each lift of the chain in Chapter 43, and gains and exceptional primes in Chapter 48. Critical pairs and the target lattice are defined at the end of this chapter.

For a prime p and a ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_p$, val is the valuation of $\mathbb{C}_p$ normalized by $\mathrm{val}(p) = 1$, with $\mathrm{val}(0) = \infty$, read after η; after an isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$ it is read through ι and the inclusion $\overline{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$. Put $\mathrm{val}_+ := \max(\mathrm{val}, 0)$. Let $\mathcal{O}_\eta$ be the ring of $z \in \mathbb{C}$ with $\mathrm{val}(\eta z) \geq 0$. For a positive integer q, let $\mathcal{O}^{[q]}$ be the ring of numbers *integral away from* q: the $z \in \mathbb{C}$ with $\mathrm{val}(\eta z) \geq 0$ for every prime $\ell \nmid q$ and every ring homomorphism $\eta\colon \mathbb{C} \to \mathbb{C}_\ell$.

34.1. Chosen places, the group $\bar{G}$ and the idempotent e_0

A finite place w of E is *chosen* if it divides π_M; let $\mathcal{M}$ be the set of chosen places and $\mathcal{M}_i$ those above p_i $(i = 0, 1)$. For $w \in \mathcal{M}$ let k_w be the residue field, and put $\bar{G} := \prod_{w\in\mathcal{M}} \mathrm{GL}_r(k_w) = \bar{G}_0 \times \bar{G}_1$, with $\bar{G}_i := \prod_{w\in\mathcal{M}_i} \mathrm{GL}_r(k_w)$. A *label* of degree r over k_w is a monic squarefree polynomial of degree r with nonzero constant term; after fixing an injective character $\overline{k_w}^\times \to \mathbb{C}^\times$ for each residue field, Green's construction associates to the label an irreducible character Ξ_c of $\mathrm{GL}_r(k_w)$ and the central idempotent $e_c := (\Xi_c(1)/\#\mathrm{GL}_r(k_w)) \sum_g \overline{\Xi_c(g)}\, g$. A family of labels $(c_w)_{w\in\mathcal{M}}$ is *good* if, for $i = 0, 1$, no c_w with $w \in \mathcal{M}_i$ has a root $z \in k_w$ with $z^{q_2} = 1$, and at least one c_w with $w \in \mathcal{M}_i$ is irreducible with a root in $\overline{k_w}$ of order $(\#k_w)^r - 1$. Let $e_0 \in \mathbb{C}[\bar{G}]$ be the sum, over the good families, of the tensor products $\bigotimes_w e_{c_w}$; it is a central idempotent.

The larger finite group $\prod_{w\in\mathcal{M}} \mathrm{GL}_{rt}(k_w)$, with matrices of size $rt = 5r^2$, has order $\mathcal{N} := \prod_{w\in\mathcal{M}} \#\mathrm{GL}_{rt}(k_w)$.

Problem 34.1.1. For $i = 0, 1$ there are exactly f chosen places above p_i, and $\#k_w = p_i$ for $w \in \mathcal{M}_i$. Moreover $\log \mathcal{N} \leq (rt)^2 f \log M \leq x^{32}$.

34.2. The gap and the weights of the lifts

The *gap* is $\Delta := 2r^8$. For $j \geq 0$ and $\tau \in \Phi$ the *weight of lift* j is $\kappa_j(\tau) := \kappa(\tau) + (j + \frac{1}{2})\Delta$. The rank-$r$ target uses the following initial segment ς_τ of the

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target parameter at τ:

$$\varsigma_\tau := \Big(\tfrac{k}{2}-1,\ \tfrac{k}{2}-2,\ \ldots,\ 1-\tfrac{k}{2};\ \ -\kappa(\tau)-\Delta,\ -\kappa(\tau)-2\Delta,\ \ldots,\ -\kappa(\tau)-m\Delta\Big),$$

with $k-1$ entries before the semicolon and m after it (k is even). The full sequence continues with $-\kappa(\tau)-(m+1)\Delta, -\kappa(\tau)-(m+2)\Delta, \ldots$.

Problem 34.2.1. The entries of ς_τ are strictly decreasing; its j-th entry after the semicolon is $\Delta/2-\kappa_j(\tau)$.

Definition 34.2.1. The *diagram* of the data assigns to $\tau \in \Phi$ and an integer $h \geq 1$ the number $\beta(\tau,h) \geq 0$ of its columns of height h at τ:

- $\beta(\tau,1) := a$ if $\tau \in \Phi_a$, and $\beta(\tau,1) := 0$ otherwise;
- $\beta(\tau,k) := \kappa_1(\tau) - \lfloor (k+t+1)/2 \rfloor$;
- $\beta(\tau,h) := \Delta-(t+1)$ if $h > k$ and $t+1$ divides $h-k$;
- $\beta(\tau,h) := 0$ for every other h.

The first two cases are disjoint since $k \geq 2$ (Problem 33.3.2). Both differences are positive for the data of Chapter 33. The forms on X_j have, at τ, the highest weight read from β at the heights below n_j: its entry p, numbered from 0 for $0 \leq p < n_j$, is $\sum_{p<h<n_j} \beta(\tau,h)$ plus the last entry, which is 0 for $j=0$ and $\lfloor n_{j-1}/2 \rfloor - \kappa_j(\tau)$ for $j \geq 1$.

Problem 34.2.2. At $\tau \in \Phi_a$ the columns of height below k are precisely the a columns of height one, and at $\tau \notin \Phi_a$ there are no columns of height below k. The corresponding highest weights on X_0 are $(a,0,\ldots,0)$ and 0, respectively, and their Weyl modules are the modules A_τ of Section 34.5. Formally, for $k=4$, $t=5$, $\Delta=8$, $a=3$ and $\kappa_j(\tau) = 12+8j$ at a place $\tau \in \Phi_a$ (values that no data have), the weight on X_1 at τ is $(0,-3,-3,-3)$ followed by six entries -18, and that on X_2 is $(-3,-6,-6,-6)$ followed by six entries -21 and six entries -23.

34.3. The spaces

Each space has a Gram matrix J over E and an adelic frame, chosen in the proof, in which J becomes the specified diagonal sign matrix at infinity and the identity at the finite places; this frame is used for the rational subgroup and for all lattice and reduction conditions. The *standard lattice stabilizer* $K^\circ(V)$ of such a framed space V of rank n is the group of finite adelic unitary matrices of V that are integral at every finite place, the stabilizer of the standard lattice of integral finite adelic vectors of length n. For $j \geq 0$, X_j is the standard positive definite space of rank n_j. The spaces Y_j, $0 \leq j \leq m$, are those of Section 33.3; for $j < m$ the identity block of Y_j is X_j, and Y_m is identified, through a chosen bijection of bases, with the orthogonal complement in X_m of its last basis vector. For $1 \leq j \leq m$, number the r basis vectors of W_j from 0 to $r-1$; its Gram matrix is diagonal, with sign $+1$ at index $i < k+j-1$ and -1 at the other indices; and V_j is the standard positive definite space of rank $r+t$.

Problem 34.3.1. W_j has exactly $m-j$ negative signs for $1 \leq j \leq m$. Formally, for $r=9$, $m=4$, $j=1$, the negative basis vectors are the last three.

34.4. Principal forms, good forms and the target

Let V be such a framed space of rank r, with numbered signs, such as Y_0 or W_j; for Y_0 the sign paired with entry i, numbered from 0, is $(-1)^{\max(i+1-k,0)}$. A *principal form* on V is a bounded continuous complex function on its adelic unitary group, invariant on the left under the rational points and on the right under the *principal level* of integral finite adelic unitary matrices that reduce to the identity at every $w \in \mathcal{M}$, such that, if the product e over τ of the discrete-series kernels with doubled parameter $2\varsigma_\tau$ and the numbered signs of V is integrable, right convolution by e fixes the form. They form a space $H(V)$, on which $\bar{G}$ acts by right translation through integral lifts. The *good forms* are $H_g(V) := e_0 H(V)$. The *target forms* and their dimension are $T := H_g(Y_0)$ and $d_T := \dim_{\mathbb{C}} T$.

34.5. Source forms

The *source forms* S are the algebraic modular forms on the unitary group of X_0 with values in $\bigotimes_{\tau\in\Phi} A_\tau$, where A_τ is the Weyl module of highest weight $(a, 0, \dots, 0)$ for $\tau \in \Phi_a$ and $A_\tau = \mathbb{C}$ otherwise. A form is evaluated on the finite adelic unitary group, and a rational element acts on its values through the entrywise conjugates of its archimedean components. The level consists of the integral finite adelic unitary matrices reducing to the identity at every $w \in \mathcal{M}$ and having last row congruent to $(0, \dots, 0, 1)$ modulo π_N. Put $D_X := \dim_{\mathbb{C}} S$. For each prime $w \mid \pi_N$ of E fix an integral finite idele z_w of norm one, equal to 1 outside w and $\bar{w}$, whose reduction at w has order $N-1$; right translation by the scalar z_w is an operator T_w on S. A source form φ is *ramified* if $\sum_{i=0}^{N-2} T_w^i \varphi = 0$ for every $w \mid \pi_N$; the ramified forms are a subspace S_{ram}. Write $S_{\mathrm{ram}}^{\mathrm{int}}$ for the ramified forms whose gauge of Chapter 48 is at most 1 at every finite adelic argument, for every prime ℓ and every field isomorphism $\iota\colon \mathbb{C} \to \overline{\mathbb{Q}}_\ell$, composed with the inclusion into $\mathbb{C}_\ell$.

34.6. Tame characters and twists

A *tame character* ξ consists of a continuous unitary character of the norm-one ideles of E, trivial on the norm-one elements of $E^\times$, a finite set Q_ξ of finite places of E, and characters ψ_w of $k_w^\times$ for all finite places w, such that: ξ has weight 0 and tame part (ψ_w), that is, $\xi(u) = \prod_w \psi_w(u_w \bmod w)$ for every norm-one idele u with integral finite part; $\psi_w = 1$ for $w \notin Q_\xi$; $\bar{w} \notin Q_\xi$ for $w \in Q_\xi$; and $\mathrm{Nm}\, Q_\xi := \prod_{w\in Q_\xi} \#k_w$ is prime to MN. A *twist* ϑ is a tame character with $\psi_w \neq 1$ for every $w \in Q_\vartheta$; it is *m-fold* if $\psi_w^n \neq 1$ for every $w \in Q_\vartheta$ and $1 \le n \le m$. The trivial twist 1 has $Q = \varnothing$. For a nonnegative integer n, the power ξ^n as a tame character has the same set Q_ξ and tame part (ψ_w^n).

Problem 34.6.1. A character ψ of $k_w^\times$ has $\psi^n \neq 1$ for $1 \le n \le m$ if and only if its order exceeds m. For $k_w = \mathbb{F}_{13}$, exactly 8 of the 12 characters have order exceeding 3.

34.7. The characters of the lifts

The proof fixes two characters of the norm-one ideles, trivial on the rational points: $\mu^\flat$, of weight $-\kappa_0$ with trivial tame part, and χ_M, of weight 0, whose tame part (ω_w) is trivial off $\mathcal{M}$ and has order q_2 at each $w \in \mathcal{M}$. With υ the splitting character of the Weil representation restricted to the norm-one ideles (weight 1,

trivial tame part), the *character of lift* j is $\mu_j^0 := \mu^\flat\,(\upsilon^{-\Delta}\chi_M)^j$, of weight $-\kappa_j$ and tame part (ω_w^j).

34.8. The integral vacuum

For each s the proof fixes $c_{\mathrm{vac}} \in \mathbb{C}^\times$ such that c_{vac} times the vacuum of the standard rank-one space over E_s has values integral under every ring map $\mathbb{C} \to \mathbb{C}_p$, for every prime p, at all finite arguments, and all conjugates over K of these values have absolute value at most $\exp(v_0 q_1^2)$, with v_0 as in Section 33.3. The *integral vacuum* of a space of rank n is c_{vac}^n times its vacuum.

34.9. The chain and the single lifts

The weight on Y_m at τ has the same last entry as the weight on X_m. Put $\beta^\circ(h) := \Delta/2 - \lfloor (t+1)/2 \rfloor$ if $h \geq k$ and $t+1$ divides $h-k$, and $\beta^\circ(h) := 0$ otherwise; for $1 \leq h < n_m - 1$ the weight on Y_m has $\beta_Y(\tau, h) := \beta^\circ(h) + \beta(\tau, h+1) - \beta^\circ(h+1)$ columns of height h, that is, its entry p, numbered from 0 for $0 \leq p < n_m - 1$, is $\sum_{p<h<n_m-1} \beta_Y(\tau, h)$ plus the last entry.

For a tame character ξ, the *lower chain* Θ_ξ is a linear map from $H(Y_0)$ to algebraic forms of this weight on Y_m: the composite of m normalized theta lifts $Y_{j-1} \to Y_j$, $1 \leq j \leq m$, whose j-th step is built from the character $\mu_j^0 \xi^j$, followed by the passage to algebraic forms on Y_m. For an m-fold twist ϑ, Θ_ϑ also denotes its restriction to T, with target the subspace of this weight on which the final level acts by the character prescribed by ϑ, and $Z(\vartheta) := \det(\Theta_\vartheta^* \Theta_\vartheta)$. For $1 \leq j \leq m$, the *single lift on principal forms* $\hat{u}_{\xi,j}$ is the theta lift from $H(W_j)$ to the algebraic level forms on V_j of the weight, level and character of the descent pair for (ξ, j), built from $\mu_j^0 \xi$, and the *single lift* $u_{\xi,j}$ is its restriction to $H_g(W_j)$; for an m-fold twist ϑ, $Z_j(\vartheta) := \det(u^* u)$ with $u := u_{\vartheta^j,j}$. Chapter 43 describes the datum, characters and scalar in each step of the lower chain; the final level and its character are named in the remark below. Both determinants use the Petersson inner product on the source and the class-sum inner product on the algebraic level forms in the target.

34.10. Critical pairs

For a continuous unitary character μ of the norm-one ideles of E, trivial on the norm-one elements of $E^\times$, and a finite set S of finite places of E, write $\bar{S}$ for the set of conjugate places.

Definition 34.10.1. The *partial Hecke value* $L^S(z, \mu)$, for real $z > 1$, is the Euler product over the finite places $v \notin S \cup \bar{S}$ of $(1 - c_v(\#k_v)^{-z})^{-1}$, where c_v is the value of the Hecke character $x \mapsto \mu(x/\bar{x})$ of E at a chosen uniformizer at v. At $z = 1$ it is the limit from the right, where this limit exists and is finite. For $z > 1$, if $S \cup \bar{S}$ contains the places where this Hecke character ramifies, the product is its L-function without the Euler factors at $S \cup \bar{S}$.

A *critical pair* $P = (\nu, z)$ consists of such a character ν, of weight $-\mathrm{wt}_P$ with $\mathrm{wt}_P \colon \Phi \to \mathbb{Z}_{\geq 0}$ (the *negative weight* of P), residue characters ψ_w that give its tame part and are trivial off $\mathcal{M}$, and an integer z with $1 \leq z \leq t$ and $z \leq \mathrm{wt}_P(\sigma) \leq 4b$ for all $\sigma \in \Phi$. It is *ramified* if $\psi_w \neq 1$ exactly for $w \in \mathcal{M}$. For a tame character ξ and $\epsilon = \pm 1$ its *ratio* is

$$R_P(\xi, \epsilon) := \frac{|L^{\mathcal{M} \cup Q_\xi}(z, (\nu \xi^\epsilon)^{-1})|}{|L^{\mathcal{M}}(z, \nu^{-1})|}, \qquad R_P(\xi) := R_P(\xi, 1).$$

Problem 34.10.1. For $1 \le j \le m$, a critical pair with tame part (ω_w^j), the j-th power of the tame part of χ_M, is ramified. Formally, if a family of residue characters is trivial off the nonempty set $\mathcal{M}$ and has order 6 at every place of $\mathcal{M}$, then for exactly 9 of the integers $1 \le j \le 10$ its j-th power is nontrivial at every place of $\mathcal{M}$.

34.11. The target lattice

Recall the isomorphisms $\iota \mid \ell$ of Chapter 5. For an m-fold twist ϑ, the target of Θ_ϑ is the space of algebraic forms on Y_m of the weight specified in Section 34.9 on which the final level acts by the character prescribed by ϑ, with the class-sum inner product. A form F in it is *integral at* ι if its values have gauge (Chapter 48) at most 1 at every finite adelic argument, for the composite of ι with the inclusion $\overline{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$.

Definition 34.11.1. The *target lattice* is the following twisted lattice on T, in the sense of Chapter 6, Section 6.1. Its twists are the m-fold twists, with the trivial twist 1; its maps are the lower chains Θ_ϑ on T; for every $n \in \mathbb{N}$, the twist ϑ is *allowed* at n if $\gcd(n, \operatorname{Nm} Q_\vartheta) = 1$; and $\mathfrak{L}_{\vartheta,\iota}$ is the $\mathcal{O}_\iota$-module of forms in the target of Θ_ϑ that are integral at ι. The coordinates of a form F in the target of Θ_1 are the coordinates, in the standard basis of the Weyl module, of its values $F(g_c)$ at fixed representatives g_c of the classes at its level; their rational Gram matrix is block diagonal, with one copy of the Gram matrix of the Weyl module for each class. For $v \in T$ and $\sigma \in \operatorname{Aut}(\mathbb{C})$, $\|v\|_\sigma^2$ is the squared length, for this Gram matrix, of the coordinate vector of $\Theta_1 v$ with σ applied to each coordinate. A vector $v \in T$ is *integral at* $\iota \mid \ell$ if $\Theta_\vartheta v \in \mathfrak{L}_{\vartheta,\iota}$ for every ϑ allowed at ℓ. A family $y_1, \dots, y_n$ in T has *covolume* at most c if for every $\sigma \in \operatorname{Aut}(\mathbb{C})$ the determinant of the products of the coordinate vectors of the $\Theta_1 y_i$, with σ applied, taken by the same Gram matrix, has absolute value at most e^c. It has *index* at most δ if there is a function $e\colon \mathbb{N} \to \mathbb{R}_{\ge 0}$ satisfying both conditions below. For every finite set U of primes, $\sum_{\ell \in U} e(\ell) \log \ell \le \delta$. For every prime ℓ and every $\iota \mid \ell$, there is a matrix $u \in M_n(\mathcal{O}_\iota)$ with $|\det u|_\iota \ge \ell^{-e(\ell)}$ such that $\sum_k u_{ik} y_k$ is integral at ι for every i.

These are the index and covolume of Section 6.2 for this twisted lattice. The proof shows that these data satisfy the pairing condition of a twisted lattice and that the maps Θ_ϑ are injective on T (Problem 59.4.3); by definition its volume at ϑ is $Z(\vartheta)$. The stabilizers of the classes at the levels of the targets are trivial, so $\|v\|_\sigma^2$ is also the weighted sum of Chapter 42, Section 42.3, for the form $\Theta_1 v$.

In the Lean formalization. In the Lean proof [GL] (`grh/src/`, with the namespace `New.` omitted below): The objects of this chapter are `Data.IsChosen`, `Data.chosenGL`, `Data.e`$_0$, `Data.gap`, `Data.liftWeight`, `Data.targetParameter`, `Data.spaceY`, `Data.spaceW`, `Data.descentPair`, `Data.principalForms`, `Data.goodForms`, `Data.targetForms`, `Data.SourceForm`, `Data.ramifiedSourceForms`, `Data.integralRamified`, `Data.TameChar`, `Data.Twist`, `Data.MFoldTwist`, `Data.untwistedLiftChar`, `Parameters.vacuumScale`, `Parameters.intVacuum`, `Data.lowerChain`, `Data.`Θ, `Data.goodLift`, `Data.chainDeterminant` and `Data.liftDeterminant`; step j of the normalized chain is `Data.liftY` at index $j-1$. For a tame character, `Data.lowerChainAtLevel` gives `Data.lowerChain` its codomain of forms of the prescribed level and character, given by `Data.lastLevel` and the pullback of

`Data.lastLevelChar` under reduction to `Data.lastResidueLevel`; the weight on Y_m is `Data.weightY`, with columns `Data.columnsY`, given by `Data.weightY_eq`. The definitions of W_j, of the descent pair (with the same standard target space V_j at every j), of both single lifts and of $Z_j(\vartheta)$ accept every nonnegative integer j; both single lifts are zero outside $1 \le j \le m$. For an m-fold twist, `Data.`Θ restricts `Data.lowerChainAtLevel` to T. The single lift on principal forms is `Data.singleLift`, and its restriction to good forms is `Data.goodLift`. The constant v_0 of Chapter 33 is `Parameters.vacuumBoundConst`; the characters $\mu^\flat$, χ_M, (ω_w) are `Data.chiZero`, `Data.chiM`, `Data.chiMLocal`. The properties of e_0, of these characters, of μ_j^0 and of c_{vac} stated in Sections 34.1, 34.7 and 34.8 are `Data.isCentralProj_e`$_0$, `Data.chiZero_hasWeight`, `Data.chiM_hasWeight`, `Data.orderOf_chiMLocal`, `Data.chiMLocal_eq_one_of_not_isChosen`, `Data.untwistedLiftChar_hasWeight` and `Parameters.isVacuumScale_vacuumScale`; the restricted splitting character υ is `WeilData.upsilonU1`, of weight 1 and trivial tame part by `WeilData.upsilonU1_hasWeight`. Problem 34.2.1 is `Data.targetParameter_eq` and `Data.targetParameter_strictAnti`. The truncated valuation val_+ on $\overline{\mathbb{Q}}_\ell$ is `ellVal`, and $\mathcal{O}^{[q]}$ is `Seesaw.integralAway` q, which enters the clause on the coefficients of `Data.IsChainDatum`. The group $\bar{G}$ is `Data.chosenGL`, a product over the pairs (i, w) with $w \in \mathcal{M}_i$, so $\bar{G}_i$ is its factor at i; $\mathcal{N}$ is `Data.chosenGroupOrder`. The place count in Problem 34.1.1 is `Data.card_chosenPlace`, and the residue-field size is the restriction of `Data.q_eq_levelPrime` to chosen places. The endpoint of the bound is `Data.log_chosenGroupOrder_le`; its proof gives the intermediate bound as well. Definition 34.2.1 is `Data.columnCount`; the last entry is `Data.lastEntry`, and the weight on X_j is `Data.weightX`, given through the diagram by `Data.weightX_eq`. $K^\circ(V)$ is `stdLevel`. Definition 34.10.1 is `AutChar.L`, `Data.CriticalPair` (with wt_P its field `negWeight`), `Data.CriticalPair.IsRamified` and `Data.CriticalPair.ratio`; at $z = 1$, `AutChar.L` takes the value of `extendFrom` from $(1, \infty)$. The first assertion of Problem 34.10.1 uses `Data.chiMLocal_zpow_ne_one_iff_isChosen`, stated for all integers $n \neq 0$ with $|n| \le 8m$. The count for order 6 is a formal example. Definition 34.11.1 is `Data.targetLattice` (in `New/Reduction/IntegralTargetForms.lean`), a `TwistedLattice`, with `integralVectors`, `IndexLe`, `CovolumeLe` and `normSqAt`; its volume and injectivity are `Data.volume_targetLattice` and `Data.injective_lift_targetLattice`; the pairing condition is its field `norm_inner_le_one`; triviality of the stabilizers is `Data.classAut_lastResidueLevel_eq_bot`, and `Data.normSqAt_targetLattice` compares the two norms.

Hints

§34.1.

34.1.1: $p_i \equiv 1 \pmod{q_1 d}$ splits completely in E (Problem 33.2.3), and the ideal (π_M) of $\mathcal{O}_K$ is the product of one prime above p_0 and one above p_1; $[E : K] = f$. Use $\#\mathrm{GL}_n(\mathbb{F}_q) < q^{n^2}$ for $n \ge 1$, $r \le 3x^5$, Problems 33.2.2, 33.3.3 and 33.5.4, $\varphi(q_0) \le C$ and $(\log b)^{100} \le x$.

§34.2.

34.2.1: $\kappa(\tau) + \Delta > k/2$.

34.2.2: $n_0 = k$. Below $n_1 = 10$ the nonzero columns have heights 1 and 4; below $n_2 = 16$ also 10.

§34.3.

34.3.1: $r - (k + j - 1)$.

§34.6.

34.6.1: $\mathbb{F}_{13}^{\times}$ is cyclic of order 12.

§34.10.

34.10.1: ω_w has order q_2 at $w \in \mathcal{M}$ and is trivial off $\mathcal{M}$; $q_2 \geq 2^s > m$ by Definition 33.3.1(2).

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.