DENOMINATORS OF PROJECTIONS TO LIFTS

Prerequisites. Discrete valuation rings and lengths of modules; the elementary divisor theorem. The Chebotarev density theorem and the Brauer–Nesbitt theorem [CR, (30.16)]. From Chapter 29, *Congruence modules*: the congruence module $\mathrm{C}(\,\cdot\,;\,\cdot\,)$ and Problems 29.1.1 and 29.1.2. From Chapter 30, *Eisenstein congruences and Selmer classes*: Problem 30.2.2 and Ribet's lemma, Theorem 30.2.1. From Chapter 24, *Hecke operators and theta lifts*: Section 24.4, with Problems 24.4.2 and 24.4.3. From Chapter 32: Section 32.3 on $X_0(11)$ and, for Section 65.4, the Weil data, the eigenforms and the attached representations of Section 32.4. For Section 65.3, Chapters 27 and 28 on Tate twists and odd classes; Chapter 31 is background. For Section 65.4, the data of Chapter 33; the single lifts $u_{\vartheta,j}$, the good forms and the twists of Chapter 34, whose notation we use; Definitions 61.3.2 and 61.3.4; ℓ-adic representations of Galois groups, crystalline representations and their Hodge–Tate weights, at the level of definitions.

To compare lifts at the trivial twist with lifts at a twist chosen for each $\iota \mid \ell$, one needs to control the denominators of the orthogonal projections of integral test forms onto spaces of lifts. How large can they be? In Chapter 29 the exponent of a congruence module gave the scalar denominator of a projection onto an eigenspace, and in Chapter 30 a congruence produced a Galois cohomology class. Here we measure a denominator by the determinant of an operator that clears it. Sections 65.1 and 65.2 are models, the second the smallest example, a class set with two elements; Section 65.3 describes the Galois side; Section 65.4 gives the definitions and the statement. Theorem 65.4.1 is read from the proof [GL]; of its bound, the local loss is developed in Chapter 66.

65.1. Clearing a projection

Let $\mathcal{O}$ be a discrete valuation ring with fraction field K and valuation ord, normalized by $\mathrm{ord}(\varpi) = 1$ for a uniformizer ϖ; we use $\mathrm{ord}(0) = +\infty$. Let V be a finite-dimensional K-vector space with a lattice L, and let $V = \mathcal{F} \oplus \mathcal{F}'$ with projection $\mathrm{pr}\colon V \to \mathcal{F}$ along $\mathcal{F}'$. Write $\mathrm{C}(L;\mathcal{F}) := \mathrm{pr}(L)/(L \cap \mathcal{F})$, as in Chapter 29. An operator $C_\ell \in \mathrm{End}_K(\mathcal{F})$ (the subscript recalls the prime ℓ of the later sections) *clears* pr on a subset $Y \subseteq L$ if $C_\ell(\mathrm{pr}(y)) \in L$ for every $y \in Y$. A scalar c clears pr on L when $c\,\mathrm{pr}(L) \subseteq L$; an operator may clear different directions of $\mathcal{F}$ by different amounts, and we measure it by $\mathrm{ord}(\det C_\ell)$.

Problem 65.1.1. C_ℓ clears pr on L if and only if $C_\ell(\mathrm{pr}(L)) \subseteq L \cap \mathcal{F}$. The least value of $\mathrm{ord}(\det C_\ell)$ over such C_ℓ is $\mathrm{len}_{\mathcal{O}}\,\mathrm{C}(L;\mathcal{F})$.

The operators that clear pr on a given Y form an $\mathcal{O}$-module; the definitions make sense for any valuation ring $\mathcal{O}$ and any $\mathcal{O}$-submodule L of V. The next problem combines clearing operators that are diagonal in one basis.

Problem 65.1.2. Let $\mathcal{O}$ be a valuation ring with infinite residue field and valuation ord, such as the ring of integers of $\bar{\mathbb{Q}}_\ell$ with $\mathrm{ord}(\ell) = 1$, and let $b_1, \dots, b_n$ be a

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basis of $\mathcal{F}$. Let $C_\ell^{(1)}, \dots, C_\ell^{(n)}$ clear pr on Y, with $C_\ell^{(j)} b_i = \lambda_{ji} b_i$. Then some $C_\ell = \sum_j u_j C_\ell^{(j)}$ with $u_j \in \mathcal{O}$ has $C_\ell b_i = \mu_i b_i$ with $\operatorname{ord}(\mu_i) \le \operatorname{ord}(\lambda_{ii})$ for every i; it clears pr on Y.

Now let K be a subfield of $\mathbb{C}$ stable under complex conjugation, and $\mathcal{O}$ a discrete valuation ring with fraction field K. Let V carry a positive definite hermitian form $\langle\,,\rangle$, linear in the first variable, let $\mathcal{F}' = \mathcal{F}^\perp$, and let T be a self-adjoint operator on V with $T(L) \subseteq L$, diagonalizable with eigenvalues in $\mathcal{O}$, such that $\mathcal{F} \subseteq \ker(T - \lambda)$. The vectors of $\mathcal{F}^\perp$ with the eigenvalue of $\mathcal{F}$,

$$F_T := \mathcal{F}^\perp \cap \ker(T - \lambda),$$

are the *false lifts* for T. Let $\Pi(X) := \prod_\mu (X - \mu)$, the product over the eigenvalues $\mu \ne \lambda$ of T, each taken once.

Problem 65.1.3. For $y \in L$ orthogonal to F_T, $\Pi(T)\, y = \Pi(\lambda)\, \mathrm{pr}(y)$. Hence the scalar $\Pi(\lambda)$ clears pr on the $y \in L$ orthogonal to F_T, and the determinant of the scalar $\Pi(\lambda)$ acting on $\mathcal{F}$ has valuation $\dim \mathcal{F} \cdot \operatorname{ord}(\Pi(\lambda))$.

The cost $\operatorname{ord}(\Pi(\lambda)) = \sum_\mu \operatorname{ord}(\lambda - \mu)$ counts congruences between the eigenvalue of $\mathcal{F}$ and the other eigenvalues. The restriction to vectors orthogonal to the false lifts cannot be dropped.

Problem 65.1.4. Let $V = K^2$ with $\langle x, y\rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2$, $L = \mathcal{O}^2$, $T = \lambda \cdot 1$, $\mathcal{F} = K(1,1)$, and suppose $\operatorname{ord}(2) > 0$. Then $F_T = K(1,-1)$, no unit $c \in \mathcal{O}^\times$ clears pr on L, and $c = 1$ clears it on the $y \in L$ orthogonal to F_T.

65.2. Two classes

We take the class set of Section 24.4 of Chapter 24: the two left ideal classes of a maximal order in the quaternion algebra over $\mathbb{Q}$ ramified at 11 and ∞, indexed by their right orders, with $w_1 = 2$ and $w_2 = 3$. Functions on the two classes are vectors $\varphi = (\varphi(1), \varphi(2))$, with

$$\langle \varphi, \varphi' \rangle := \frac{\varphi(1)\overline{\varphi'(1)}}{w_1} + \frac{\varphi(2)\overline{\varphi'(2)}}{w_2},$$

and the Brandt matrix $B(\ell)$, for a prime $\ell \ne 11$, acts by $(B(\ell)\varphi)(i) = \sum_j B_{ij}(\ell)\varphi(j)$. Recall from that section that $B(\ell)$ has integer entries, that $w_j B_{ij}(\ell) = w_i B_{ji}(\ell)$ and $\sum_j B_{ij}(\ell) = \ell + 1$, and that $B(\ell)$ is the action of T_ℓ on $M_2(\Gamma_0(11))$, which is spanned by an Eisenstein series of T_ℓ-eigenvalue $\ell + 1$ and by the form called f there; we write

$$f_{11} := q \prod_{n \ge 1} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n \ge 1} a_n q^n.$$

By Problem 24.4.2, $B(2) = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $B(3) = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$. The constant function 1 plays the role of the lifts: $\mathcal{F}$ is the line of constant functions, and pr is the orthogonal projection onto $\mathcal{F}$. The integral functions are $L := \mathbb{Z}^2$. To apply Section 65.1 over $\mathbb{Z}_{(\ell)}$, use the lattice $\mathbb{Z}_{(\ell)}^2$ in $\mathbb{Q}^2$. Over $\mathbb{Q}_\ell$ use the lattice $\mathbb{Z}_\ell^2$ and the same bilinear formula without the bar.

Problem 65.2.1. $B(\ell)$ is self-adjoint for $\langle\,,\rangle$, $B(\ell)1 = (\ell+1)1$, and $\varphi_{11} := (2, -3)$ spans $\mathcal{F}^\perp$ and satisfies $B(\ell)\varphi_{11} = a_\ell \varphi_{11}$.

Problem 65.2.2. $\mathrm{pr}(\varphi) = \frac{1}{5}\big(3\varphi(1) + 2\varphi(2)\big) \cdot 1$, and $5\,\mathrm{pr} = B(2) + 2 = B(3) + 1$. Hence $\mathrm{pr}(L) = \frac{1}{5}\mathbb{Z} \cdot 1$ and $\mathrm{C}(L;\mathcal{F}) \cong \mathbb{Z}/5\mathbb{Z}$. The least value of $\mathrm{val}_5(\det C_\ell)$ for C_ℓ clearing pr on $\mathbb{Z}_5^2$ is 1; for $\ell \neq 5$ the identity clears pr on $\mathbb{Z}_\ell^2$. The scalar $\Pi(\lambda)$ of Problem 65.1.3 is 5 both for $T = B(2)$ and for $T = B(3)$.

What congruence do the Brandt eigenvalues satisfy?

Problem 65.2.3. $B(\ell) - a_\ell = (\ell + 1 - a_\ell)\mathrm{pr}$. Hence $a_\ell \equiv 1 + \ell \pmod 5$ for every prime $\ell \neq 11$; for example $a_{13} = 4$ (Problem 32.3.1).

The denominator 5 is the numerator of the mass $1/w_1 + 1/w_2 = (11-1)/12 = 5/6$. The next problem gives the denominator at primes dividing none of the weights.

Problem 65.2.4. Let I be a nonempty finite set with positive integers w_i, let $\langle \varphi, \varphi' \rangle := \sum_i \varphi(i)\varphi'(i)/w_i$ on $\mathbb{Q}_\ell^I$, let $\mathcal{F}$ be the line of constant functions, and let ℓ be a prime dividing no w_i. Then $\mathfrak{m} := \sum_i 1/w_i$ has $\mathrm{val}_\ell(\mathfrak{m}) \geq 0$, and $\mathrm{C}(\mathbb{Z}_\ell^I;\mathcal{F}) \cong \mathbb{Z}/\ell^{\mathrm{val}_\ell(\mathfrak{m})}\mathbb{Z}$.

For the quaternion algebra ramified at a prime p and ∞ the mass is $(p-1)/12$, and Mazur showed that the index of his Eisenstein ideal is the numerator of $(p-1)/12$ [Ma]. There is also an integrality question for the pairing of a cleared form with an integral form.

Problem 65.2.5. In the setting of Problem 65.2.4, drop the condition on ℓ, and let n_0 be a common multiple of the w_i. If C_ℓ clears pr on $Y \subseteq \mathbb{Z}_\ell^I$, then so does $n_0 C_\ell$, and $\langle n_0 C_\ell \mathrm{pr}(y), \varphi' \rangle \in \mathbb{Z}_\ell$ for $y \in Y$ and $\varphi' \in \mathbb{Z}_\ell^I$. In the class set with two elements, $6\langle \varphi, \varphi' \rangle \in \mathbb{Z}$ for $\varphi, \varphi' \in \mathbb{Z}^2$.

65.3. Lifts and the classes they produce

Where does the congruence of Problem 65.2.3 come from? The constant function carries the Galois representation $1 \oplus \chi_{\mathrm{cyc}}$, where χ_{cyc} is the 5-adic cyclotomic character; its trace at an arithmetic Frobenius Frob_ℓ is $1 + \ell$ for $\ell \neq 5$. The form f_{11} carries the representation ρ_{11} of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the rational 5-adic Tate module $T_5(X_0(11)) \otimes \mathbb{Q}_5$, whose trace at Frob_ℓ is a_ℓ for $\ell \nmid 55$ (Section 32.3 of Chapter 32). Let ω be the reduction of χ_{cyc} modulo 5.

Problem 65.3.1. The semisimplification of the reduction of ρ_{11} modulo 5 is $1 \oplus \omega$, and $\omega(c) = -1$ for every complex conjugation c.

The representation ρ_{11} is irreducible: it is semisimple and its commutant is $\mathbb{Q}_5$, by Faltings' theorem [Fa], since $X_0(11)$ has no complex multiplication (its j-invariant $-122023936/161051$ is not an integer). Ribet's lemma, Theorem 30.2.1, then gives a nonzero class with values in $\mathbb{F}_5(\omega)$ or $\mathbb{F}_5(\omega^{-1})$, an odd character.

In the general setting a single theta lift takes values in forms on a standard definite unitary space for the CM extension E/E^+. For a main summand R for a pair $\mathfrak{O}_0$ and a family $\mathfrak{s}$, take a pair $\mathfrak{O}$ with the same basis index sets for its two spaces and with character that of $\mathfrak{O}_0$ times a tame character ξ. The normalization S of any representation attached to a lift eigenform for $\mathfrak{s}$ on $\mathfrak{O}$ satisfies $\mathrm{tr}\, S(g) = \hat{\xi}(g)\,\mathrm{tr}\, R(g) + \sum_{0 \leq e < t'} \chi_{\mathrm{cyc}}(g)^{-e}$ for every $g \in \mathrm{Gal}(\bar{\mathbb{Q}}/E)$. Here t' is the rank difference, χ_{cyc} is the ℓ-adic cyclotomic character, and $\hat{\xi}$ is the inverse avatar of ξ (Definitions 65.4.5 and 65.4.8). The constant function above has this shape, with no R and two characters, after a twist by the inverse 5-adic cyclotomic character.

For an extension of $\chi_{\mathrm{cyc}}^{-e'}$ by χ_{cyc}^{-e} modulo ℓ^ν, the extension cocycle takes values in $(\mathcal{O}/\ell^\nu)(-a)$, where $a = e - e'$. On cocycles z of $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ a complex conjugation c acts by $z^c(g) := z(cgc)$; call z *odd* if $z^c = -z$ and *even* if $z^c = z$.

Problem 65.3.2. Let ℓ be odd and $\mathcal{O}$ the ring of integers of a finite extension of $\mathbb{Q}_\ell$. Every cocycle of $\mathrm{Gal}(\bar{\mathbb{Q}}/E^+)$ with values in $(\mathcal{O}/\ell^\nu)(-a)$ differs by a coboundary from one with $z(c) = 0$. For such z, the restriction to $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ satisfies $z^c = (-1)^a z$.

For odd ℓ and odd a, restriction from $\mathrm{Gal}(\bar{\mathbb{Q}}/E^+)$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ therefore gives classes with odd representatives. For the CM field $E \subseteq \mathbb{Q}(\mu_{q_1 d})$ of Chapter 33, let $d_{q_1 d}$ be the exponent of $(\mathbb{Z}/(q_1 d)\mathbb{Z})^\times$. Theorem 27.3.2, proved using Gauss sums as in Chapter 28, gives one positive integer X_{odd} that works simultaneously for every prime ℓ, every ring of integers $\mathcal{O}$ of a finite extension of $\mathbb{Q}_\ell$ containing a primitive $d_{q_1 d}$-th root of unity, every integer $0 \le a < t$ and every integer $\nu \ge 0$. For $a > 0$ the $\mathcal{O}$-length of the module of continuous odd cocycles of $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$ with values in $(\mathcal{O}/\ell^\nu)(-a)$, unramified off ℓ and zero on the kernel of χ_{cyc} in every decomposition group above ℓ, is at most $\mathrm{ord}_{\mathcal{O}}(X_{\mathrm{odd}})$, with a uniformizer of order 1. For $a = 0$ the same bound holds for the everywhere unramified continuous homomorphisms with values in $\mathcal{O}/\ell^\nu$. Definition 65.4.1 contains no order of a group of even classes. A class can be tested by its values at Frobenius elements; here is an example, for a Kummer class.

Problem 65.3.3. Let ℓ be an odd prime, $u \in \mathbb{Q}^\times$ not an ℓ-th power, and kum_u the Kummer class of u in $H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mu_\ell)$. For a prime $q \equiv 1 \pmod{\ell}$ with $\mathrm{val}_q(u) = 0$, kum_u is unramified at q, and $\mathrm{kum}_u(\mathrm{Frob}_q) = 0$, a condition independent of the choice of Frob_q, if and only if $u^{(q-1)/\ell} \equiv 1 \pmod{q}$. There are infinitely many such q with $\mathrm{kum}_u(\mathrm{Frob}_q) \neq 0$. For $\ell = 3$ and $u = 2$, $q = 7$ detects kum_2 and $q = 31$ does not.

Ribet's lemma needs an element on which the two characters differ (Problem 30.2.2). For the characters χ_{cyc}^{-e}, $0 \le e < t$, with χ_{cyc} the ℓ-adic cyclotomic character, the cost of an element g is the valuations of the differences $\chi_{\mathrm{cyc}}(g)^a - 1$, $0 < a < t$. If $\chi_{\mathrm{cyc}}(g)$ reduces to a generator of $\mathbb{F}_\ell^\times$ and $t \le \ell - 1$, these valuations are 0, by Problem 65.3.4. If g is an arithmetic Frobenius element at a chosen place w not dividing ℓ (Section 34.1), then $\chi_{\mathrm{cyc}}(g) = \#k_w$ and these valuations are at most $\mathrm{val}_\ell(\mathcal{N})$, by Problem 65.3.5 with $n = rt$: the integer $\mathcal{N}$ of that section is a multiple of $\#\mathrm{GL}_{rt}(k_w)$.

Problem 65.3.4. Let $g \in \mathbb{Z}_\ell^\times$ reduce to a generator of $\mathbb{F}_\ell^\times$. Then $g^a - 1 \in \mathbb{Z}_\ell^\times$ for $0 < a < \ell - 1$.

Problem 65.3.5. Let q be a prime power and $1 \le a \le n$. Then $q^a - 1$ divides $\#\mathrm{GL}_n(\mathbb{F}_q) = q^{n(n-1)/2} \prod_{i=1}^n (q^i - 1)$.

65.4. The statement

We use the data of Chapter 33: the CM field E with f infinite places, the rank r, $t = 5r$, the numbers m and k, the level primes p_0, p_1 with $M = p_0 p_1$, the level N, the moduli q_1 and d, and the budget B of Definition 33.4.1. From Chapter 34 we use the chosen places $\mathcal{M}$, labels, the good idempotent e_0 and the integer $\mathcal{N}$ (Section 34.1), the gap $\Delta = 2r^8$ and the weights κ_j of the lifts (Section 34.2), tame

characters ξ and twists ϑ with their sets Q_ξ, Q_ϑ (Section 34.6), and the single lifts $u_{\vartheta,j}$ on good forms (Section 34.9).

Fix a prime ℓ and a ring isomorphism $\iota\colon \mathbb{C} \simeq \bar{\mathbb{Q}}_\ell$. As in Chapter 34, val is the valuation of $\bar{\mathbb{Q}}_\ell$ with $\mathrm{val}(\ell) = 1$ and $\mathrm{val}(0) = \infty$, read through ι on complex numbers, and $\mathrm{val}_+ := \max(\mathrm{val}, 0)$. A form y is *integral at* ι if for every finite adelic point g the gauge of $y(g)$ at g is at most 1. To compute this gauge, use ι followed by $\bar{\mathbb{Q}}_\ell \subset \mathbb{C}_\ell$ on the weight coordinates, apply g^{-1} in the representation of that weight, and take the largest ℓ-adic absolute value of a resulting coordinate. At an infinite place τ with chosen CM-type embedding Φ_τ, the adelic action is read through $x \mapsto \iota(\Phi_\tau(\bar{x}))$ at the induced ℓ-adic place. The field E is embedded in $\bar{\mathbb{Q}}$, and $G_E := \mathrm{Gal}(\bar{\mathbb{Q}}/E)$; with c the complex conjugation of $\bar{\mathbb{Q}}$ fixed in the proof, $g^c := cgc$ for $g \in G_E$; and χ_{cyc} is the ℓ-adic cyclotomic character. For a place of $\bar{\mathbb{Q}}$ above a finite place w of E not dividing ℓ, s_w denotes its distinguished generator of tame inertia; it depends on the place of $\bar{\mathbb{Q}}$.

A *coefficient ring* $\mathcal{O}$ is the ring of integers of a finite Galois extension of $\mathbb{Q}_\ell$ in $\bar{\mathbb{Q}}_\ell$, and a statement holds *for all large* $\mathcal{O}$ if it holds for all coefficient rings containing a fixed one. The *normalized length* of an $\mathcal{O}$-module of finite length is its length divided by the ramification index $e_\mathcal{O}$ of $\mathcal{O}$ over $\mathbb{Z}_\ell$; so $\mathcal{O}/\ell$ has normalized length 1, and the normalized length does not change when $\mathcal{O}$ is enlarged and the module extended. An $\mathcal{O}$-module of infinite length has normalized length $+\infty$.

Definition 65.4.1. Let $n_\ell := \mathrm{val}_\ell(\mathcal{N})$ and $o_\ell := \mathrm{val}_\ell(X_{\mathrm{odd}})$, where X_{odd} is the integer that the proof fixes for the length bound of Section 65.3, with $0 < X_{\mathrm{odd}} \le 2^{4(q_1 d+t+3)^{27}} h_E$ and h_E the class number of E. Let $\mathrm{loc}_\ell = 4f\,\mathrm{defect}_\ell$ be the local loss of Definition 66.1.1, and put

$$\mathrm{tw}_\ell := (4f+3)\,r\,n_\ell + 2^{-\ell}, \qquad \varepsilon_\ell := t\,o_\ell + 8ft\,n_\ell + 2n_\ell + \big((4t+1)(\Delta/2)^{22} + \Delta/2\big)n_\ell.$$

The *clearing bound* is $\mathrm{cl}_\ell := \mathrm{loc}_\ell + (1+2t)\,\mathrm{tw}_\ell + \varepsilon_\ell + n_\ell$.

Definition 65.4.2. A *descent pair* $\mathfrak{D}$ has two finitely indexed families of finite places of E, chosen places and twist places; all their entries are distinct, and none is the conjugate of any entry. It has a hermitian space W of rank r' with signs on its basis indices and a specified adelic frame in which its Gram matrix is the diagonal matrix of the signs at every infinite place and the identity at every finite place, a natural number assigned to each index, an injection of those indices into the basis indices of the standard definite space $V = I_{n'}$ with $r' < n'$ (its image is the set of *selected* indices), column multiplicities $k^{\pm}(\tau, i)$ at each infinite place τ and height $i \ge 0$, and a continuous unitary character μ of $U(1)(\mathbb{A})$, trivial on rational points, with specified integral weight at each infinite place and specified tame part, a character ψ_w of $k_w^\times$ at each finite place w, trivial off the two families. Write K_0 for the stabilizer of the standard integral lattice of V in the finite adelic group. The *level* of $\mathfrak{D}$ consists of the elements of K_0 whose reductions fix the selected basis vectors at the chosen places and preserve the span of the other basis vectors at the twist places. Its *level character* is the product over the twist places w of ψ_w^{-1} applied to the determinant of the block on the selected indices. Its forms $\mathcal{A}(\mathfrak{D})$ are the algebraic forms on V on which the level acts through the level character, of the weight with column multiplicities $k^+(\tau, i) + k^-(\tau, n' - i)$ at τ and each height $0 \le i < n'$, and with determinant exponent $-\sum_{1 \le i < n'} k^-(\tau, i)$ at τ. We write $t' := n' - r'$ for its rank difference.

Definition 65.4.3. Let e_1 be an element of the complex group algebra of $\prod_w \mathrm{GL}_{r'}(k_w)$ over the chosen places of $\mathfrak{O}$, and $A \subseteq \mathcal{A}(\mathfrak{O})$ a subspace. A tuple $g = (g_w)_w$ of this group acts on $\mathcal{A}(\mathfrak{O})$ by right translation by an element of K_0 that reduces to g_w on the selected block and to the identity on the other block at each chosen place w, and to the identity at the twist places, multiplied by $\prod_w \psi_w(\det g_w)$; write $\mathrm{im}(e_1)$ for the image of the operator by which e_1 then acts. In this sheet υ_E is the splitting character of the Weil data on the ideles of E (Section 32.4), and υ is its restriction to the norm-one ideles, as in Section 34.7. For a character a of the norm-one ideles and a natural number $N_0 \geq 0$, the *Frobenius value* of a in rank N_0 at a finite place w is $q_w^{N_0/2}$ times the value, at the finite idele with the chosen uniformizer at w and 1 elsewhere, of the Hecke character $x \mapsto a(x/\bar{x})\,\upsilon_E(x)^{-N_0}$; here $q_w = \#k_w$. For a split place w such that neither w nor $\bar{w}$ is a place of $\mathfrak{O}$, the *false lifts* at w are the forms in $\mathrm{im}(e_1) \cap A$ killed by the Hecke polynomial operator of $\mathfrak{O}$ at w at the value $\big(\nu_w q_w^{n'-r'-1}\big)^{-1}$ expected of a lift, where ν_w is the Frobenius value at w of υ^{r_-} in rank r', with r_- the number of negative signs of W. A form is *orthogonal to the false lifts* for $(\mathfrak{O}, e_1, A)$ if it is orthogonal to the false lifts at every such w.

Definition 65.4.4. Let e_1 be as in Definition 65.4.3, $\mathcal{F} \subseteq \mathcal{A}(\mathfrak{O})$ a subspace, with orthogonal projection pr, and $\mathrm{den} \in \mathbb{R}$. The projection onto $\mathcal{F}$ has *denominator at most* den at ι if some operator C_ℓ on $\mathcal{F}$ has $|\iota(\det C_\ell)|_\ell \geq \exp(-\dim \mathcal{F} \cdot r' \cdot \mathrm{den})$ and, for every form y integral at ι and orthogonal to the false lifts for $(\mathfrak{O}, e_1, \mathcal{F}^\perp)$, $C_\ell(\mathrm{pr}(y))$ is integral at ι and $|\iota\langle C_\ell(\mathrm{pr}(y)), y'\rangle|_\ell \leq 1$ for every form y' integral at every ring map $\mathbb{C} \to \mathbb{C}_\ell$. Here $\langle\,,\rangle$ is the class sum of the pointwise Gram inner products, linear in the first variable; each term is weighted by the reciprocal of the order of the stabilizer of its double-coset representative.

For a twist ϑ and $1 \leq j \leq m$, take the j-th descent pair of ϑ, $e_1 = e_0$ and $\mathcal{F}$ the image of $u_{\vartheta,j}$. Then the false lifts are those of Definition 61.3.2, and the condition of Definition 65.4.4, without the bound on the pairings $\langle C_\ell(\mathrm{pr}(y)), y'\rangle$, says that den clears the projection at (ι, ϑ, j) in the sense of Definition 61.3.4.

Definition 65.4.5. The *inverse avatar* of a tame character ξ is the ℓ-adic avatar $\hat{\xi}\colon G_E \to \bar{\mathbb{Q}}_\ell^\times$, through ι, of the base change of ξ^{-1} to E: the continuous character with $\hat{\xi}(\mathrm{Art}\, x) = \iota\big(\xi(x/\bar{x})\big)^{-1}$ for every finite idele x of E, where Art is the Artin map. We use the Artin map of the proof, with values in the topological abelianization G_E^{ab} and with a local uniformizer mapping to geometric Frobenius modulo inertia. A character of G_E is evaluated there through the induced character. Let $\mathrm{ram} \in [0, \infty]$. A twist ϑ is *auxiliary with ramification at most* ram (at ι) if:

(a) every place of Q_ϑ has prime norm, and neither it nor its conjugate divides ℓMN (its conjugate is not in Q_ϑ, as for every tame character);
(b) $\hat{\vartheta}$ is locally constant, and there is a character η of the finite ideles of E, with values in $\bar{\mathbb{Q}}_\ell^\times$, such that $\hat{\vartheta}(\mathrm{Art}\, x) = \eta(x/\bar{x})$ for every finite idele x, and η is trivial on $E^\times$ and on every idele that is a unit at the places not above ℓ and is congruent to 1 at the places of Q_ϑ;
(c) for some choice of a place of $\bar{\mathbb{Q}}$ above each $w \in Q_\vartheta$, with tame generators s_w, $\sum_{w \in Q_\vartheta} \mathrm{val}_+\big(\hat{\vartheta}(s_w) - 1\big) \leq \mathrm{ram}$.

Definition 65.4.6. Let r' and t' be natural numbers. A family $\mathrm{HT} = (\mathrm{HT}_\sigma)$ of finite multisets of integers, one for each embedding $\sigma\colon E \to \mathbb{C}$, is *admissible for* (r', t') if:

(i) $1 \le r' \le r$ and $t' \le t$;
(ii) there is one $0 \le j \le m$ such that, for every σ, every $u \in \mathrm{HT}_\sigma$ satisfies $\Delta/4 \le |u| \le 2m\Delta$ or $|s_\sigma u + \kappa_j([\sigma])| \le t$; here $[\sigma]$ is the infinite place of σ, and $s_\sigma = \pm 1$ is the sign of σ for the CM type, that is, $\sigma(\sqrt{-d}) = s_\sigma\sqrt{-d}$;
(iii) $k \cdot \#\{u \in \mathrm{HT}_\sigma : |u| \le 2(\Delta/2)^2\} \le r'm$ for every σ, the members counted with multiplicity;
(iv) HT_σ and $\mathrm{HT}_{\bar\sigma}$ together have at most r' negative members, for every σ.

The members of HT_σ are called *jumps*.

Definition 65.4.7. Let $t' \ge 0$ be an integer, HT a family as in Definition 65.4.6, and $R\colon G_E \to \mathrm{GL}_{r'}(\bar{\mathbb{Q}}_\ell)$ a homomorphism. A *lattice of R over $\mathcal{O}$* is a homomorphism $G_E \to \mathrm{GL}_{r'}(\mathcal{O})$ conjugate to R by an element of $\mathrm{GL}_{r'}(\bar{\mathbb{Q}}_\ell)$. The representation R is *main-summand-like for* (t', HT) if:

(a) R is continuous, the trace pairing on the span of its image is nondegenerate, and $\operatorname{tr} R(g^c) = \chi_{\mathrm{cyc}}(g)^{-\max(t'-1,0)} \operatorname{tr} R(g^{-1})$ for every $g \in G_E$;
(b) at every place A of $\bar{\mathbb{Q}}$ above every finite place w of E: if $w \nmid \ell$, R is trivial on the elements of the inertia group in a neighbourhood of 1; if $w \mid \ell$, the restriction of R to the decomposition group is potentially crystalline for every local datum at A, that is, every embedding e of $\bar{\mathbb{Q}}$ in $\bar{\mathbb{Q}}_\ell$ inducing A, with the induced embedding of its decomposition group into $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$, and for every such datum and every $g \in \mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ its Hodge–Tate jumps at the coefficient embedding induced by g are, with multiplicity, the multiset HT_σ for $\sigma = \iota^{-1} \circ g^{-1} \circ e|_E$; and if $w \nmid M$, R is unramified ($w \nmid \ell$), respectively crystalline for every local datum ($w \mid \ell$);
(c) HT is admissible for (r', t');
(d) at every place of $\bar{\mathbb{Q}}$ above every $w \mid M$ with $w \nmid \ell$ there are an element δ of the inertia group and a natural number n_w prime to the residue characteristic such that $R(s^{n_w}) = 1$ for every inertia element s there, δ is the tame generator s_w if $w \in \mathcal{M}$, and $\operatorname{val}_+ \det(R(\delta) - 1) \le r'n_\ell$, $\operatorname{val}_+ \det\bigl(R((\delta^c)^{-1}) - 1\bigr) \le r'n_\ell$;
(e) for all large $\mathcal{O}$, the image of every lattice of R over $\mathcal{O}$ spans $M_{r'}(\mathcal{O})$ over $\mathcal{O}$; and for all large $\mathcal{O}$ and every lattice of R over $\mathcal{O}$, some $g \in G_E$ with $\chi_{\mathrm{cyc}}(g) \in \{p_0^{r'}, p_1^{r'}\}$ acts on it by a scalar;
(f) some element of an inertia group above a place dividing M and not dividing ℓ acts through R by a scalar y with $|y^{\ell^a j} - 1|_\ell = 1$ for all integers $a \ge 0$ and $1 \le j \le 4r' + 2$.

The Hodge–Tate jumps are taken in the convention of the proof, in which the cyclotomic character has jump -1.

Definition 65.4.8. Let $\mathfrak{s} = (\mathfrak{s}(w, i))$ be a family of complex numbers indexed by the finite places w of E and the integers $i \ge 0$. A form $y \in \mathcal{A}(\mathfrak{O})$ *lifts* $\mathfrak{s}$ if, at every split place w such that neither w nor $\bar w$ is a place of $\mathfrak{O}$, it is spherical and for every $T \in \mathbb{C}$ the Hecke polynomial operator at w at the value T sends y to y multiplied

by

$$\prod_{0\le i<t'} \bigl(1-q_w^i\,\nu_W T\bigr)\cdot H_{\mathfrak{s}(w)}\bigl(q_w^{t'}\nu_W T/\nu_V\bigr), \qquad H_{\mathfrak{s}(w)}(T) := \sum_{i=0}^{r'} (-T)^i q_w^{i(i-1)/2}\,\mathfrak{s}(w,i),$$

where ν_W and ν_V are the Frobenius values at w of υ^{r_-} in rank r' and of $\mu\,\upsilon^{\lfloor n'/2\rfloor}$ in rank n', respectively, as in Definition 65.4.3. A homomorphism $R\colon G_E \to \mathrm{GL}_{r'}(\bar{\mathbb{Q}}_\ell)$ is a *main summand* for a descent pair $\mathfrak{O}_0$ of ranks $r' < n'$ and $\mathfrak{s}$ if it is main-summand-like for $(n'-r', \mathrm{HT})$ for some HT, and

$$\operatorname{tr} S(g) = \hat{\xi}(g)\operatorname{tr} R(g) + \sum_{0\le e<n'-r'} \chi_{\mathrm{cyc}}(g)^{-e} \qquad (g\in G_E)$$

for every tame character ξ, every descent pair $\mathfrak{O}$ with the same basis index sets for W and for V as $\mathfrak{O}_0$ and with character μ that of $\mathfrak{O}_0$ times ξ, every form in $\mathcal{A}(\mathfrak{O})$ lifting $\mathfrak{s}$, every family P of polynomials for which that form is an eigenform, and every representation attached to P (Section 32.4). Here S, the *normalization* of that representation for $\mathfrak{O}$, is its twist by the ℓ-adic avatar, through ι, of the Hecke character $x \mapsto \upsilon^{-r_-}(x/\bar{x})\,\upsilon_E(x)^{r'}\,|x|^{r'/2}$, with r_- taken for $\mathfrak{O}$ and $|x|$ the idelic norm.

Definition 65.4.9. Let $R\colon G_E \to \mathrm{GL}_{r'}(\bar{\mathbb{Q}}_\ell)$ be a homomorphism, ξ a tame character, $g_*\in G_E$ and $\mathrm{sel}\in[0,\infty]$. Let L be a lattice of R over a coefficient ring $\mathcal{O}$ containing the values of $\hat{\xi}$, and L_{tw} the module $\mathcal{O}^{r'}$ on which $g\in G_E$ acts by $\hat{\xi}(g)L(g)$. For each $v\mid\ell$, let G_v be the decomposition group of the chosen place above v and use the local embedding of G_v into $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ fixed in the proof. In the crystalline relations below, an integral crystalline cocycle means a function $z_0\colon G_v\to L_{\mathrm{tw}}$ for which some $b_0\in B_{\max}\otimes_{\mathbb{Z}_\ell} L_{\mathrm{tw}}$ satisfies $1\otimes z_0(g) = gb_0 - b_0$ for every $g\in G_v$. Here $B_{\max}$ is the crystalline period ring of the proof, and G_v acts on the tensor product through its local embedding on $B_{\max}$ and through L_{tw} on the second factor. Then $\mathrm{Sel}(L_{\mathrm{tw}}, g_*)$ is the $\mathcal{O}$-module of locally constant maps $z\colon G_E\to L_{\mathrm{tw}}\otimes_{\mathcal{O}}(\mathrm{Frac}(\mathcal{O})/\mathcal{O})$ such that: $z(gh) = z(g) + g\,z(h)$ for all g,h; z vanishes on the inertia groups above the finite places of E that neither divide ℓM nor lie in $Q_\xi\cup\bar{Q}_\xi$; z satisfies the *crystalline relations*, that is, for each $v\mid\ell$ and every finite family of $g_i\in G_v$ and $a_i\in M_{r'}(\mathcal{O})$ such that $\sum_i a_i\,z_0(g_i)=0$ for every integral crystalline cocycle $z_0\colon G_v\to L_{\mathrm{tw}}$, also $\sum_i a_i\,z(g_i)=0$; and $z(g_*)=0$. The twist $\hat{\xi}R$ *has Selmer bound* sel if there are an element g_* of an inertia group above a place of E not dividing ℓ and an integer ζ of $\bar{\mathbb{Q}}_\ell$ such that, for all large $\mathcal{O}$, there is a lattice L of R over $\mathcal{O}$ for which $\zeta\in\mathcal{O}$, $\zeta-1\in\mathcal{O}^\times$, g_* acts on L_{tw} by the scalar ζ, and $\mathrm{Sel}(L_{\mathrm{tw}}, g_*)$ has normalized length at most sel.

Definition 65.4.10. A descent pair $\mathfrak{O}$ with a tuple of polynomials over the residue fields of its chosen places is in *lift setting* for ϑ if its indexed chosen and twist places agree with those of the data and of ϑ; its tame part is the j-th power of the tame part of χ_M from Section 34.7, times the tame part of ϑ, for some $1\le j\le m$; r' and $n'-r'$ are odd, with $r'\le r$ and $n'-r'\le t$; each polynomial is a label of degree r' (Section 34.1), and none of its roots becomes 1 on multiplication by the element of $k_w^\times$ naming the inverse tame character of $\mathfrak{O}$ at its place w; and above each p_i one polynomial is primitive with $r'=r$, or equal to $X-1$ with $r'=1$.

Definition 65.4.11. Let ϑ be a twist, $\mathfrak{O}$ a descent pair, e_1 as in Definition 65.4.3, $\mathcal{F} \subseteq \mathcal{A}(\mathfrak{O})$ a subspace and $\mathrm{sel} \in [0,\infty]$. A *lift generator* for $(\vartheta, \mathfrak{O}, e_1, \mathcal{F})$ with Selmer bound sel is a form $y \in \mathcal{F}$ such that:

(a) y lies in the range of the idempotent of a tuple of polynomials with which $\mathfrak{O}$ is in lift setting for ϑ (Definition 65.4.10), and the whole range of this idempotent is contained in $\mathrm{im}(e_1)$;

(b) y lifts a family $\mathfrak{s}$ for which there are a descent pair $\mathfrak{O}_0$ with the same basis index sets for W and for V as $\mathfrak{O}$, whose character μ times ϑ is that of $\mathfrak{O}$, and a main summand R for $\mathfrak{O}_0$ and $\mathfrak{s}$ such that $\hat{\vartheta}R$ has Selmer bound sel (Definition 65.4.9).

Theorem 65.4.1. *Let ℓ and ι be as above, and let ϑ be a twist that is auxiliary with ramification at most tw_ℓ. Let $\mathfrak{O}$ be a descent pair of ranks $r' < n'$, e_1 an element of the complex group algebra of $\prod_w \mathrm{GL}_{r'}(k_w)$ over its chosen places, and $\mathcal{F} \subseteq \mathcal{A}(\mathfrak{O})$ a subspace contained in the span of the lift generators for $(\vartheta, \mathfrak{O}, e_1, \mathcal{F})$ with Selmer bound tw_ℓ. Then the projection onto $\mathcal{F}$ has denominator at most $\mathrm{cl}_\ell \log \ell$ at ι, in the sense of Definition 65.4.4.*

Status: read from the proof [GL]; not proved here. The next three problems are arithmetic of the clearing bound and of determinants.

Problem 65.4.1. Let r', t' be integers with $1 \le r' \le r$ and $0 \le t' \le t$, and let $0 \le \mathrm{sel}, \mathrm{ram} \le \mathrm{tw}_\ell$. Then

$$\mathrm{sel} + 2t'\,\mathrm{ram} + r'\,\mathrm{loc}_\ell + \varepsilon_\ell + n_\ell \le r'\mathrm{cl}_\ell.$$

Problem 65.4.2. Let $\mathcal{F}$ be a finite-dimensional complex vector space, $n_0 \ge 0$ an integer, $u_0 \ge 0$, and $C_\ell \in \mathrm{End}_{\mathbb{C}}(\mathcal{F})$ with $\mathrm{val}(n_0\iota(\mu)) \le u_0$ for every eigenvalue μ of C_ℓ. Then, with $|\ell|_\ell = \ell^{-1}$, $|\iota(\det(n_0 C_\ell))|_\ell \ge \exp(-\dim \mathcal{F} \cdot u_0 \log \ell)$.

Problem 65.4.3. The sum of $2^{-\ell}\log\ell$ over all primes ℓ is $0.3776\ldots$, and for every finite set P_0 of primes, $\sum_{\ell \in P_0} \mathrm{tw}_\ell \log \ell \le (4f+3)\, r \log \mathcal{N} + 0.378$.

The clearing bounds of all primes together fit the budget B: the proof bounds by B the sum in Problem 65.4.4. Let C_1 be the number of $U(1)$ double cosets at the chosen level. In Chapter 67 the allowance for the pairing on the rank-one lifts is $(\mathrm{cl}_\ell + \mathrm{val}_\ell(C_1))\log\ell$ (Section 67.2). With x the length of the prime sum and b the weight scale of Chapter 33, consider the following five comparisons.

$$\log\bigl(2^{4(q_1 d + t + 3)^{27}} h_E\bigr) \le x^{2000}, \qquad (4f+3)\, r \log \mathcal{N} + 2 \le x^{999}, \qquad \log C_1 \le x^{30},$$

$$t\, x^{2000} + \bigl(8ft + 2 + (4t+1)(\Delta/2)^{22} + \Delta/2\bigr)\log \mathcal{N} \le x^{4000},$$

$$4 \cdot 10^6 e^{180} f\, \frac{m}{k}\, b \log b + (1+2t)\, x^{999} + x^{4000} + \log \mathcal{N} + x^{30} \le B.$$

Problem 65.4.4. Granting the five displayed inequalities, for every finite set P_0 of primes

$$\sum_{\ell \in P_0} \bigl(\mathrm{cl}_\ell + \mathrm{val}_\ell(C_1)\bigr)\log\ell \;\le\; 4{\cdot}10^6 e^{180} f\, \frac{m}{k}\, b \log b + (1+2t)\, x^{999} + x^{4000} + \log\mathcal{N} + x^{30} \;\le\; B.$$

By Theorem 61.3.1(b), for a twist ϑ of modulus prime to ℓ, a real $\mathrm{cl}(\ell)$ that clears the projection at (ι, ϑ, j) bounds the ℓ-adic norm, through ι, of the determinant of the compression of a comparison operator to the image of $u_{\vartheta,j}$ by $\exp(\dim_{\mathbb{C}} H_g(W_j) \cdot r \cdot \mathrm{cl}(\ell))$. For $1 \le j \le m$, $e_1 = e_0$ and $\mathcal{F}$ the image of $u_{\vartheta,j}$, so that $r' = r$, the proof

deduces from Theorem 65.4.1 that every real $\mathrm{cl}(\ell) \geq \mathrm{cl}_\ell \log \ell$ clears the projection at (ι, ϑ, j) in the sense of Definition 61.3.4, provided ϑ is auxiliary with ramification at most tw_ℓ and every eigensystem occurring in a nonzero good form on W_m has a main summand, for the j-th descent pair of the trivial twist, whose twist by $\hat{\vartheta}$ has Selmer bound tw_ℓ. Chapter 67 chooses an m-fold twist θ_ι for each $\iota \mid \ell$, and at index j one takes $\vartheta = \theta_\iota^j$. Of the terms of cl_ℓ, Chapter 66 develops the local loss loc_ℓ and proves the bound for its sum (Theorem 66.4.1); Theorem 65.4.1 itself is taken from the proof.

In the Lean formalization. In the Lean proof [GL] (`grh/src/`, with the namespace `New.` omitted), Definition 65.4.1 is `Selmer.clearingBound`, with X_{odd} the integer `Selmer.oddClassBound`; Definitions 65.4.2–65.4.4 are `DescentPair`, `DescentPair.falseLiftsAt` (the value at the chosen uniformizer is `HeckeChar.atUniformizer`) and `Selmer.DenominatorLe`; Definitions 65.4.5–65.4.11 are `Selmer.IsAuxTwist` for `Selmer.badPlaces`, Q_ϑ and $\hat{\vartheta}$, with `Data.TameChar.invAvatar` and its formula `Selmer.invAvatar_eq_of_artinMap_eq` for the Artin map `artinMap`, `Selmer.IsMainSummandWeights`, `Selmer.IsMainSummandLike` (the jumps are those of `Period.jump`), `Selmer.IsMainSummand` (its eigenform and attachment conditions are `IsEigenform` and `GaloisRep.Attached`), the existential closure in g_*, ζ of `Selmer.HasDivisibleSelmerLengthLe` with modulus Q_ξ, `Selmer.IsLiftSetting` and `Selmer.IsLiftGenerator`. The module in Definition 65.4.9 has ordinary length `Selmer.crystallineSelmerLength` and uses `Selmer.crysRelations`; its local embedding and period ring are those of `Selmer.localChart` and `Period.crysCocycles`. Theorem 65.4.1 is `Selmer.denominatorLe_clearingBound`, and the good-lift case is `Selmer.denominatorLe_range_goodLift`, followed by `Selmer.DenominatorLe.clearsProjection` for the passage to $\mathrm{cl}(\ell) \geq \mathrm{cl}_\ell \log \ell$. The two inequalities of Problem 65.4.4 occur in the calculation of `Selmer.sum_clearingBound_add_padicValNat_le`, which states the bound by B for all admissible data and all finite sets of primes.

Hints

§65.1.

65.1.1: C_ℓ maps $\mathcal{F}$ to $\mathcal{F}$. For the minimum, $\mathrm{pr}(L)$ and $L \cap \mathcal{F}$ are lattices in $\mathcal{F}$; use elementary divisors.

65.1.2: Choose the u_j generically modulo the maximal ideal.

65.1.3: Decompose y into eigenvectors of T; $\ker(T-\lambda) = \mathcal{F} \oplus F_T$ is orthogonal to the other eigenspaces.

§65.2.

65.2.1: Apply the balance and row-sum identities to the two orthogonal lines.

65.2.2: Apply the projection onto constants to the coordinate functions; then Problem 65.1.1.

65.2.3: Compare both sides on a basis.

65.2.4: Compute pr on the indicator functions.

65.2.5: Inspect each weight in the class sum.

§65.3.

65.3.1: The characteristic polynomial of $\rho_{11}(\mathrm{Frob}_\ell)$ is $X^2 - a_\ell X + \ell$ for $\ell \nmid 55$ (Section 32.3 of Chapter 32). Reduce on the lattice $T_5(X_0(11))$; ρ_{11} is

unramified outside 55. Use Problem 65.2.3, the Chebotarev density theorem and the Brauer–Nesbitt theorem for characteristic polynomials.

65.3.2: Evaluate the cocycle relation on $c\cdot c = 1$ and on $c\cdot g\cdot c$, using $\chi_{\mathrm{cyc}}(c) = -1$.

65.3.3: $\mathrm{kum}_u(\mathrm{Frob}_q)$ is read off from the action of Frob_q on $u^{1/\ell}$ modulo a prime above q. Since u is not an ℓ-th power, $[\mathbb{Q}(u^{1/\ell}) : \mathbb{Q}] = \ell$, so $\mathbb{Q}(\mu_\ell, u^{1/\ell})/\mathbb{Q}(\mu_\ell)$ has degree ℓ, as $\gcd(\ell, \ell-1) = 1$; apply the Chebotarev density theorem to its nontrivial elements.

§65.4.

65.4.1: Use $r' \geq 1$ and $t' \leq t$.

65.4.3: Use the factorization of $\mathcal{N}$. For the prime sum, sum through 31 and bound the rest by $\sum_{k>31} k\,2^{-k}$.

65.4.4: Sum the four terms of cl_ℓ separately. For loc_ℓ use Theorem 66.4.1; for tw_ℓ, Problem 65.4.3; for ε_ℓ, n_ℓ and $\mathrm{val}_\ell(C_1)$, the inequality $\sum_{\ell\in P_0} \mathrm{val}_\ell(n)\log\ell \leq \log n$ for an integer $n \geq 1$, with the bound for X_{odd} of Definition 65.4.1.

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