THE COST OF ONE LIFT

Prerequisites. From Chapter 5: heights of families (Section 5.3, Problem 5.3.2). From Chapter 6: sizes read at the trivial twist, valuations at a twist that depends on the prime. The data of Chapter 33 and the objects of Chapter 34. The factorization of Chapter 55. The statements of Chapters 61, 62 and 64 recalled in Section 60.1.

Chapter 55 reduces the Gram determinant of the chain to the determinants $Z_j(\vartheta)$ of single lifts, where for an m-fold twist ϑ the j-th lift is taken at the twist ϑ^j. The proof needs a bound for the height of $Z_j(1)/Z_j(\vartheta)$ when ϑ varies with the prime ℓ and with an isomorphism $\iota \mid \ell$. How does the proof bound the ratio without evaluating either determinant? It multiplies the Gram determinant at a twist by the determinant of the compression of a second operator to the image of the lift, chosen so that the product is a fixed constant times a power of a ratio of Hecke values. The compression determinant is bounded under the automorphisms of $\mathbb{C}$ at the trivial twist only, and at ℓ at the twist (Chapter 61). The ratio of Hecke values is a product of ratios attached to critical pairs (Chapter 62), and for a regular family the inverses of these ratios have bounded height (Chapters 63 and 64). This chapter states the inputs and deduces Theorem 60.2.1 from them. One input, Theorem 60.1.1, is read from the code; the others are proved in the four chapters just named, up to the background each of them lists.

60.1. The inputs

We use the data of Chapter 33: the number f of infinite places, the scales $x < x_1 < b$ and $a = xb$, the number m of lifts, the ranks r and k, the direction amount h_{dir} and the denominator budget $B = U/(mk) = fa\log^3 b/(mk)$ (Definition 33.4.1); and the objects of Chapter 34: twists ϑ and m-fold twists, with modulus Q_ϑ and $\operatorname{Nm} Q_\vartheta$, the target dimension d_T, the single lifts $u_{\vartheta,j}$, and $Z_j(\vartheta) = \det(u^*_{\vartheta^j,j} u_{\vartheta^j,j})$ for an m-fold twist ϑ.

Fix $1 \le j \le m$. The Hecke value product $\mathcal{H}_j(\vartheta)$ of lift j at a twist ϑ is that of Definition 61.3.1. Comparison operators (Υ_ϑ), one operator on the target space of $u_{\vartheta,j}$ for every twist ϑ, are defined in Definition 61.3.3, their compression determinants D_ϑ in (1), and the statement that a real $\mathrm{cl}(\ell)$ clears the projection at (ι, ϑ, j) in Definition 61.3.4. A family $\theta = (\theta_\iota)$ of m-fold twists, one for each prime ℓ and each $\iota \mid \ell$, is regular if it satisfies clauses (i) and (ii) of Definition 64.2.2 and clause (iii) of Definition 64.2.3. Chapter 65 treats the denominators in (iii), and Chapter 67 the existence of regular families.

Theorem 60.1.1. *For every $1 \le j \le m$ the operators $\Upsilon_\vartheta := \mathcal{N} h^2\, g_\vartheta g_\vartheta^*$, one for each twist ϑ, form a comparison operator. Here g_ϑ is a gluing map from tuples of forms on a smaller definite space to the target space of $u_{\vartheta,j}$, h is the hook number of the diagram of the weight of those forms, and $\mathcal{N}$ is the order of Section 34.1, a common multiple of the orders of the stabilizers at the levels used.*

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This theorem is read from the code and is not proved in this workbook. For clause (1) of Definition 61.3.3 the proof uses $\mathcal{H}_j(\vartheta)/\mathcal{H}_j(1) = |c_1'|^2/|c_\vartheta'|^2$, with $c_\vartheta' \neq 0$ for every twist. For (2) it also uses $g_\vartheta^* u_{\vartheta,j} = (c_\vartheta')^{-1} u^\circ$, with u° injective and independent of ϑ, as in Problem 61.1.3. For (4) and (5) it writes Υ_ϑ as a sum of operators of rank at most one over a tight frame of tuples.

Three statements are recalled. By Theorem 61.3.1, for every comparison operator every D_ϑ is nonzero,

$$|\iota(D_\vartheta)|_\ell \le e^{d_T r\,\mathrm{cl}(\ell)} \qquad \text{and} \qquad |\sigma(D_1)| \le e^{3d_T f m b \log^2 b},$$

the first for every twist ϑ, every $\iota \mid \ell$ with ℓ prime to $\operatorname{Nm} Q_\vartheta$ and every real $\mathrm{cl}(\ell)$ that clears the projection at (ι, ϑ, j), the second for every $\sigma \in \mathrm{Aut}(\mathbb{C})$; and by Problem 61.3.2, for every m-fold twist ϑ,

$$\frac{Z_j(1)}{Z_j(\vartheta)} = \frac{D_{\vartheta^j}}{D_1}\Big(\frac{\mathcal{H}_j(\vartheta^j)}{\mathcal{H}_j(1)}\Big)^{-d_T}.$$

By Problem 62.5.1, for every twist ϑ the quotient $\mathcal{H}_j(\vartheta)/\mathcal{H}_j(1)$ is a power of $\operatorname{Nm} Q_\vartheta$ times the square of a product of $k/2 + 5m$ ratios $R_P(\vartheta, \epsilon)$ of ramified critical pairs P (Definition 34.10.1), of which $k/2$ are main for j (Definition 62.4.1) and have $\epsilon = 1$. By Theorem 64.4.1, for a regular family θ and a critical pair P main for j, the family $\iota \mapsto R_P(\theta_\iota^j)^{-1}$ has height at most $6B - h_{\mathrm{dir}}$.

60.2. The deduction

Let θ be regular and $1 \le j \le m$, let (Υ_ϑ) be the comparison operator of Theorem 60.1.1, and write $\vartheta_\iota := \theta_\iota^j$.

Problem 60.2.1. The family $\iota \mapsto D_{\vartheta_\iota}/D_1$ has height at most $d_T(3fmb\log^2 b + rB)$.

Problem 60.2.2. The family $\iota \mapsto \big(\mathcal{H}_j(\vartheta_\iota)/\mathcal{H}_j(1)\big)^{-1}$ has height at most $10mx_1 + k(6B - h_{\mathrm{dir}})$.

It remains to compare the constants. The following inequality is derived here from the sizes of the data.

Problem 60.2.3. Let $f \ge 1$, $b > 1$ with $\log b \ge 16$, $x \ge 1$, $\sqrt{x} \le m \le 2\sqrt{x}$, $r \ge 2m$, $k = r + 1 - m$, $1 \le x_1 \le b$, $a = xb$ and $B = fa\log^3 b/(mk)$. Then for every real h_{dir}

$$3fmb\log^2 b + rB + 10mx_1 + k(6B - h_{\mathrm{dir}}) \le k(B\log b - h_{\mathrm{dir}}).$$

The data satisfy these hypotheses.

Problems 60.2.1, 60.2.2, 61.3.2, 64.1.2 and 60.2.3, with Problem 5.3.2, prove the following.

Theorem 60.2.1. *For the data, every regular family θ of m-fold twists, and every $1 \le j \le m$, the family*

$$\iota \longmapsto \frac{Z_j(1)}{Z_j(\theta_\iota)}$$

has height at most $d_T\, k\,(B\log b - h_{\mathrm{dir}})$.

Problem 60.2.4. For the data, $B\log b < h_{\mathrm{dir}}$. So the bound of Theorem 60.2.1 is $-d_T k(h_{\mathrm{dir}} - B\log b)$ with $h_{\mathrm{dir}} - B\log b > 0$.

So the bound of Theorem 60.2.1 is, per unit of d_T, k times the budget $B \log b$ less k times the direction amount. Chapter 54 combines these bounds for $1 \le j \le m$ with the factorization of Chapter 55 and the height bound there for its correction factor. Of the inputs, Theorem 60.1.1 is read from the code; Chapters 61, 62, 63 and 64 prove the others.

In the Lean formalization. In the Lean proof [GL], in the readable edition: Theorem 60.2.1 is `Data.heightLe_liftDeterminant_ratio` and Theorem 60.1.1 is `Data.isComparisonOperator_comparisonOp`, both in `New/OneLift/LiftVolumeHeight.lean`; comparison operators are `Data.IsComparisonOperator`, and the budget B is `Data.denominatorBudget`. The declarations behind the recalled statements are named in Chapters 61, 62 and 64.

Hints

60.2.1: Problem 64.1.1 with $H = D_1$, which is nonzero; the two bounds recalled from Theorem 61.3.1; and the reals $\mathrm{cl}(\ell)$ of clause (iii) of Definition 64.2.3, whose sums are at most B.

60.2.2: Problem 62.5.1 and Theorem 64.4.1 for the $k/2$ main pairs, clause (ii) of Definition 64.2.2 for the $5m$ others, Problems 5.3.2 and 64.1.2; by (i), $\mathrm{Nm}\, Q_{\vartheta_\iota}$ is a unit at ℓ.

60.2.3: The terms in h_{dir} cancel. Compare everything with $kB = fxb \log^3 b/m \ge fmb \log^3 b/4$, and use $r \le 2k$. For the data, use the inequalities between m, r, x, x_1 and b in Chapter 33.

60.2.4: $h_{\mathrm{dir}} \ge b \#\Phi_a$ (Problem 33.4.5), $f = \#\Phi_a\, \varphi(q_0)$ (Problem 33.2.2); compare $\varphi(q_0)\, x \log^4 b$ with mk.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.