THE DETERMINANT OF A CHAIN

Prerequisites. Linear algebra over $\mathbb{C}$: adjoints and determinants. From Chapter 23: the Gaussian moments c_a of Section 23.2, and Problems 23.2.1, 23.2.3 and 23.3.2. From Chapter 34: the chosen places $\mathcal{M}$, the spaces Y_j, W_j, V_j, the principal and good forms $H(V) \supseteq H_g(V)$, the target forms $T = H_g(Y_0)$ of dimension d_T, tame characters ξ with their sets Q_ξ, twists and m-fold twists ϑ, the characters μ_j^0 of the lifts, the lower chain Θ_ξ, the single lifts $\hat{u}_{\xi,j}$ and $u_{\xi,j}$, the determinants $Z(\vartheta)$, $Z_j(\vartheta)$, and the partial Hecke values $L^S(z,\nu)$ and critical pairs, with their ratios $R_P(\vartheta,\epsilon)$ (Definition 34.10.1). From Chapter 56: the twist factor c_Q (Definition 56.3.1), the Rallis operators $\mathcal{R}_j(\xi;V)$ (Definition 56.3.2) and Theorem 56.4.1. From Chapter 57: the Petersson product and Corollary 57.4.1. From Chapter 59: the ratios $\rho_{ij}(\vartheta)$, the product $\mathcal{R}_\vartheta$ and the relation $\doteq$ (Definitions 59.3.2, 59.4.1, 59.3.1), Theorem 59.4.1 and Problem 59.4.3. From Chapter 5: heights of families (Problem 5.3.2); from Chapter 64: families with small ratios (Definition 64.2.2). Chapter 8 explains why a chain of m maps is used; it is not needed for the problems.

How does the determinant $Z(\vartheta)$ of the chain change with the m-fold twist ϑ? By Theorem 55.4.1, its quotient by $Z(1)$ is the product of the m quotients $Z_j(\vartheta)/Z_j(1)$ for single lifts, times a correction factor: a product of pairwise ratios $\rho_{ij}(\vartheta)$ of Hecke L-values, multiplied by $(\operatorname{Nm} Q_\vartheta)^{ez/2}$ with an integer e_Z independent of ϑ, and raised to the power d_T. The problems derive Theorem 55.4.1 from three statements of the following chapters: the Gram identity of one lift (Theorem 56.4.1), the comparison of the good forms on W_j and on Y_0 (Corollary 57.4.1), and the Gram identity of the chain (Theorem 59.4.1), which rests on the exchange identity (Theorem 58.5.1). The proof also states Theorem 55.4.1. Section 55.1 treats the linear algebra in its simplest setting; Sections 55.2–55.4 assemble the theorem. Section 55.5 gives two further results used in the proof of Theorem C; both are read from the proof (Theorems 55.5.1 and 55.5.2).

55.1. Volumes of maps

For a linear map $u\colon U \to V$ of finite-dimensional complex inner product spaces let

$$Z(u) := \det_U(u^* u),$$

the determinant on the domain U; we call it the *volume* of u. Dimensions are complex dimensions, the determinant on the zero space is 1, and inner products are conjugate-linear in the first argument and linear in the second.

Problem 55.1.1. For every orthonormal basis $e_1, \dots, e_d$ of U, $Z(u) = \det\big(\langle ue_a, ue_b\rangle\big)_{a,b}$. The number $Z(u)$ is real and ≥ 0, and $Z(u) > 0$ if and only if u is injective.

Problem 55.1.2. Let S be an endomorphism of U and $a \in \mathbb{C}$ with $\langle ux, uy\rangle = a\langle x, Sy\rangle$ for all $x, y \in U$. Then $Z(u) = a^{\dim U} \det S$.

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Problem 55.1.3. If $\theta_j\colon V_{j-1}\to V_j$, for $1\le j\le m$, are linear maps between spaces of one and the same dimension, then $Z(\theta_m\circ\cdots\circ\theta_1)=\prod_{j=1}^{m}Z(\theta_j)$.

When the dimension grows along the chain, equality can fail.

Problem 55.1.4. Let $\theta_2\colon\mathbb{C}^2\to\mathbb{C}^2$ be $\operatorname{diag}(1,2)$, and let $\theta_1,\theta_1'\colon\mathbb{C}\to\mathbb{C}^2$ be $x\mapsto(x,0)$ and $x\mapsto(0,x)$. Then $Z(\theta_1)=Z(\theta_1')=1$ and $Z(\theta_2)=4$, while $Z(\theta_2\theta_1)=1$ and $Z(\theta_2\theta_1')=4$.

What replaces Problem 55.1.3 is an identity for $\Theta^*\Theta$ on the first space itself (Theorem 59.4.1); Problem 59.1.2 is a model of how such an identity can be propagated along a chain.

For $a\ge 0$ divisible by 4, the two lifts from U(1) of Chapter 23 used here have a one-dimensional source. The lift $\Theta_{(a)}$ from U(1) to U(1) is multiplication by $c_a/4$ (Problem 23.2.3), and by Problem 23.3.2 the lift $\Theta_{(a)}$ from U(1) to U(2) satisfies

$$\|\Theta_{(a)}\varphi\|^2=\lambda_a\|\varphi\|^2,\qquad \lambda_a:=\frac{1}{128}\sum_j\binom{a}{j}c_j^2c_{a-j}^2.$$

This finite calculation illustrates the inner product of a lift with itself; the strict rank inequality of the inner product formula (Theorem 56.2.1) first allows a source of rank 1 when the target has rank 3, the case with which Chapter 56 begins.

Problem 55.1.5. For $4\mid a$, the composite of the lift $\mathrm{U}(1)\to\mathrm{U}(1)$ with the lift $\mathrm{U}(1)\to\mathrm{U}(2)$ of Chapter 23, in degree a, has volume $\lambda_a c_a^2/16$, the product of the two volumes.

55.2. One lift

For $1\le j\le m$ and a tame character ξ, the single lift $\hat u_{\xi,j}$ on $H(W_j)$ and its restriction $u_{\xi,j}$ to $H_g(W_j)$ are those of Section 34.9; $H(V)$ carries the Petersson product (Section 57.3). We recall two statements. By Theorem 56.4.1, for every $1\le j\le m$ there is $a_j\in\mathbb{C}$, $a_j\neq 0$, such that for every tame character ξ,

$$\hat u_{\xi,j}^*\,\hat u_{\xi,j}=a_j\,c_{Q_\xi}\,\mathcal{R}_j(\xi;W_j)$$

on $H(W_j)$, with c_Q the twist factor of Definition 56.3.1 and $\mathcal{R}_j(\xi;V)$ the Rallis operator of Definition 56.3.2; and $\hat u_{\xi,j}$ is injective. By Corollary 57.4.1, for $1\le j\le m$: $\dim H_g(W_j)=d_T$, and for every tame character ξ, $\mathcal{R}_j(\xi;Y_0)$ maps T to itself, $\mathcal{R}_j(\xi;W_j)$ maps $H_g(W_j)$ to itself, and

$$\det\bigl(\mathcal{R}_j(\xi;W_j)\mid H_g(W_j)\bigr)=\det\bigl(\mathcal{R}_j(\xi;Y_0)\mid T\bigr).$$

Problem 55.2.1. For $1\le j\le m$ and every tame character ξ, $Z(u_{\xi,j})=(a_jc_{Q_\xi})^{d_T}\det\bigl(\mathcal{R}_j(\xi;Y_0)\mid T\bigr)$. Moreover $Z(u_{\xi,j})>0$.

55.3. Twists and the ratios

Tame characters ξ, their sets Q_ξ and powers ξ^n, twists and m-fold twists ϑ are as in Chapter 34; the power ξ^n has the same set Q_ξ.

Problem 55.3.1. If ϑ is an m-fold twist and $1\le n\le m$, then ϑ^n is a twist with the same set Q_ϑ, so $c_{Q_{\vartheta^n}}=c_{Q_\vartheta}$.

For $1 \le i < j \le m$ and an m-fold twist ϑ, $\rho_{ij}(\vartheta)$ is the ratio of Definition 59.3.2: with $\nu := \mu_j^0/\mu_i^0$,

$$\rho_{ij}(\vartheta) := \prod_{z=1}^{t} \frac{|L^{\mathcal{M}\cup Q_\vartheta}\big(z, (\nu\vartheta^{j-i})^{-1}\big)|}{|L^{\mathcal{M}}(z,\nu^{-1})|},$$

a nonnegative real number. These ratios enter the Gram identity of the chain through the exchange identity (Theorem 58.5.1), whose scalar has squared modulus a constant times the exchange factor $\mathrm{ex}_h(S,\nu)$ of Definition 58.2.3; Chapter 59 applies it with $h = t$ and $S = \mathcal{M} \cup Q_\vartheta$, once for each pair $i < j$.

Problem 55.3.2. $\rho_{ij}(1) = 1$ for $1 \le i < j \le m$.

55.4. The factorization

For an m-fold twist ϑ, the lower chain Θ_ϑ of Section 34.9, restricted to T, takes values in a space of algebraic level forms on Y_m with the class-sum inner product; it is injective (Problem 59.4.3). In the notation of Section 55.1, $Z(\vartheta) = Z(\Theta_\vartheta|_T)$ and $Z_j(\vartheta) = Z(u_{\vartheta^j,j})$. Let $\mathcal{R}_\vartheta = \mathcal{R}_1(\vartheta;Y_0)\,\mathcal{R}_2(\vartheta^2;Y_0)\cdots\mathcal{R}_m(\vartheta^m;Y_0)$ be the endomorphism of T of Definition 59.4.1. By Theorem 59.4.1 there is a function c from m-fold twists to $\mathbb{C}$ with $c \doteq \prod_{1\le i<j\le m}\rho_{ij}$, that is (Definition 59.3.1), with a real number C' and an integer e' such that $|c(\vartheta)|^2 = C'(\mathrm{Nm}\,Q_\vartheta)^{e'}\big(\prod_{1\le i<j\le m}\rho_{ij}(\vartheta)\big)^2$ for every m-fold twist ϑ, and such that for every m-fold twist ϑ and all $x, y \in T$,

$$\langle \Theta_\vartheta x, \Theta_\vartheta y\rangle = c(\vartheta)\, c_{Q_\vartheta}^m\, \langle x, \mathcal{R}_\vartheta y\rangle.$$

Problem 55.4.1. Let X be a set with a point 1, let $N\colon X \to \mathbb{R}_{>0}$ and $f\colon X \to \mathbb{R}_{\ge 0}$ with $N(1) = f(1) = 1$, and let $c\colon X \to \mathbb{C}$, $C' \in \mathbb{R}$ and $e' \in \mathbb{Z}$ satisfy $|c(x)|^2 = C'N(x)^{e'}f(x)^2$ for all x. If $c(1) \ne 0$, then $C' = |c(1)|^2$ and $|c(x)/c(1)| = N(x)^{e'/2}f(x)$ for all x.

Problem 55.4.2. In Theorem 59.4.1, if $c(1) = 0$ then $T = 0$. For every m-fold twist ϑ,

$$Z(\vartheta) = \Big(c(\vartheta)\Big/\prod_{j=1}^{m} a_j\Big)^{d_T}\prod_{j=1}^{m} Z_j(\vartheta),$$

and $Z(\vartheta) > 0$.

Problem 55.4.3. If $T \ne 0$, then for every m-fold twist ϑ,

$$\frac{Z(\vartheta)}{Z(1)} = \Big((\mathrm{Nm}\,Q_\vartheta)^{e'/2}\prod_{1\le i<j\le m}\rho_{ij}(\vartheta)\Big)^{d_T}\prod_{j=1}^{m}\frac{Z_j(\vartheta)}{Z_j(1)}.$$

If $T = 0$, both sides equal 1 (a real number to the power 0 is 1).

The proof states the normalized determinant identity as follows.

Theorem 55.4.1. *Let P be parameters and D admissible data for P. There is an integer e_Z such that for every m-fold twist ϑ:*

(a) $0 < Z(\vartheta)$;
(b) $0 < Z_j(\vartheta)$ *for every integer j with* $1 \le j \le m$;
(c) *in* $\mathbb{C}$,

$$\frac{Z(\vartheta)}{Z(1)} = \Big((\mathrm{Nm}\,Q_\vartheta)^{e_Z/2}\prod_{i=1}^{m}\prod_{j=i+1}^{m}\rho_{ij}(\vartheta)\Big)^{d_T}\prod_{j=1}^{m}\frac{Z_j(\vartheta)}{Z_j(1)}.$$

Here $Z(\vartheta)$ and $Z_j(\vartheta)$ are complex numbers by definition, and (a), (b) say that they are real and positive; the base of the power d_T in (c) is a real number, with $e_Z/2$ a real exponent, and e_Z does not depend on ϑ. Problems 55.2.1 and 55.4.1–55.4.3, with Theorem 56.4.1, Corollary 57.4.1 and Theorem 59.4.1, prove Theorem 55.4.1; one may take $e_Z = e'$ when $T \neq 0$. Thus the volume of the chain at a twist, relative to the trivial twist, is expressed through m volumes of single lifts, a power of $\operatorname{Nm} Q_\vartheta$, and $m(m-1)/2$ ratios ρ_{ij} of L-values.

55.5. Auxiliary forms

The proof of Theorem C extends a linearly independent family of target forms to a basis. For each m-fold twist ϑ, a common nonzero scalar makes the ϑ-lifts of the added forms integral at every $\iota \mid \ell$ with $\ell \nmid \operatorname{Nm} Q_\vartheta$. Both statements of this section are read from the proof. The *turn ratios* are

$$\rho'_{ij}(\vartheta) := \prod_{z=1}^{t} \frac{|L^{\mathcal{M}\cup Q_\vartheta}\big(z, (\mu^0_{i+1}\mu^0_{j+1}\,\vartheta^{j+1})^{-1}\big)|}{|L^{\mathcal{M}}\big(z, (\mu^0_{i+1}\mu^0_{j+1})^{-1}\big)|}, \qquad 0 \le i, j < m,$$

a factor with zero denominator being 0. For an automorphism σ of $\mathbb{C}$ write $\|y\|^2_\sigma$ for the squared norm of the conjugate of a form y by σ.

Theorem 55.5.1 (auxiliary forms)**.** *There are an integer e_γ and numbers $\gamma(\vartheta) \in \mathbb{C}^\times$, one for each m-fold twist, with $\gamma(1) = 1$ and*

$$|\gamma(\vartheta)| = (\operatorname{Nm} Q_\vartheta)^{e_\gamma/2} \prod_{0\le i,j<m} \rho'_{ij}(\vartheta),$$

such that T is spanned by the forms y with the two properties: $\gamma(\vartheta)\Theta_\vartheta y$ is integral at every $\iota \mid \ell$ with $\ell \nmid \operatorname{Nm} Q_\vartheta$, for every m-fold twist ϑ; and $\|\Theta_1 y\|^2_\sigma \le \exp(2x_1^2 b^2)$ for every automorphism σ of $\mathbb{C}$.

Theorem 55.5.2 (small ratios)**.** *Let θ be a family of m-fold twists with small ratios (Definition 64.2.2). For every integer e, the families*

$$\iota \mapsto \Big((\operatorname{Nm} Q_{\theta_\iota})^{e/2} \prod_{1\le i<j\le m} \rho_{ij}(\theta_\iota)\Big)^{-1} \quad \text{and} \quad \iota \mapsto \Big((\operatorname{Nm} Q_{\theta_\iota})^{e/2} \prod_{0\le i,j<m} \rho'_{ij}(\theta_\iota)\Big)^{-1}$$

have height at most $m^2 t x_1$.

By Problem 59.4.6 each ρ_{ij} is a product of t ratios $R_P(\vartheta^{j-i}, 1)$ with P a ramified critical pair; the proof shows that each ρ'_{ij} is likewise a product of t ratios $R_P(\vartheta^n, 1)$, with $1 \le n \le m$ and P a ramified critical pair.

Problem 55.5.1. Let θ be a family of m-fold twists such that, for every $1 \le n \le m$ and every ramified critical pair P, the ratios $R_P(\theta_\iota^n, 1)$ are nonzero and the family $\iota \mapsto R_P(\theta_\iota^n, 1)^{-1}$ has height at most x_1. Then the families $\iota \mapsto \big(\prod_{1\le i<j\le m} \rho_{ij}(\theta_\iota)\big)^{-1}$ and $\iota \mapsto \big(\prod_{0\le i,j<m} \rho'_{ij}(\theta_\iota)\big)^{-1}$ have height at most $m^2 t x_1$; the first is a product of $m(m-1)/2$ inverse ratios ρ_{ij}^{-1}, the second of m^2.

In the Lean formalization. In the Lean proof [GL], Theorem 55.4.1 is `New.Data.exists_chainDeterminant_pos_and_div_chainDeterminant_one_eq` in `New/Determinant/Chain/NormOfChain.lean` of the readable edition; Theorem 55.5.1 is `exists_span_isAuxiliaryForm_eq_top` in `New/Determinant/Twist/LongForms.lean`, with ρ'_{ij} the `turnRatio` of `New/Determinant/`

`AuxiliaryForms/Turns.lean`; and Theorem 55.5.2 is `HasSmallRatios.heightLe_inv_rpow_mul_prod_quotientRatio` in `NormOfChain.lean` together with `HasSmallRatios.heightLe_inv_rpow_mul_prod_turnRatio` in `LongForms.lean`.

Hints

§55.1.

55.1.1: $\langle ue_a, ue_b\rangle = \langle e_a, u^*u\, e_b\rangle$, and u^*u is positive semidefinite with kernel $\ker u$.

55.1.2: Rewrite the hypothesis as an identity of endomorphisms of U.

55.1.3: In orthonormal bases of spaces of equal dimension, $Z(u) = |\det u|^2$.

55.1.4: Problem 55.1.1 with $U = \mathbb{C}$.

55.1.5: Problem 55.1.1 on a line.

§55.2.

55.2.1: Problem 55.1.2 on $H_g(W_j)$, which $\mathcal{R}_j(\xi; W_j)$ preserves; then the equality of determinants and $\dim H_g(W_j) = d_T$. For positivity, Problem 55.1.1 and the injectivity of $\hat{u}_{\xi,j}$.

§55.3.

55.3.1: The definition of ϑ^n in Section 34.6, and Definition 56.3.1.

55.3.2: At the trivial twist $Q_1 = \varnothing$ and $1^{j-i} = 1$.

§55.4.

55.4.1: Evaluate at 1 first.

55.4.2: For the first statement, Theorem 59.4.1 at $\vartheta = 1$, and injectivity. For the formula, Problems 55.1.2, 55.3.1 and 55.2.1; compare the factors c_Q for ϑ^j and ϑ.

55.4.3: Take absolute values in Problem 55.4.2 and divide by the case $\vartheta = 1$, using Problems 55.4.1 and 55.3.2. For $T = 0$, Corollary 57.4.1.

§55.5.

55.5.1: For a product, add the upper bounds for the heights of its factors (Problem 5.3.2). A height bound may be increased, and $m(m-1)/2 \le m^2$.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.