THE SEESAW IDENTITY

Prerequisites. From Chapter 21: the lattices I_n over $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$, the groups Γ_L, forms of weight (a) and their norm (Sections 21.1 and 21.3), and Problems 21.1.2 and 21.1.4. From Chapter 23: the lifts $\Theta_{(a)}$, the moments c_a, and Problems 23.1.4, 23.1.5, 23.2.1, 23.2.3, 23.2.4, 23.2.5, 23.4.2 and 23.5.1. From Chapter 36: the finite Poincaré map of Section 36.3, the Poincaré map $\mathcal{P}$, the maps A_s, and Theorem 36.4.2. From Chapter 33: the data and the spaces, forms and twists of Chapter 34, in the notation fixed there. Inner products of forms on definite lattices are linear in the first variable.

In the proof, Theorem A concerns lifts of the normalized translates A_s along the chain $Y_0 \to \cdots \to Y_m$, whose first space is indefinite, while the estimates of size and divisibility are proved along the chain $X_0 \to \cdots \to X_m$ of definite spaces. How does the proof pass between the chains? For each twist, the seesaw identity expresses the lift along the first chain of the Poincaré image of a ramified form φ as one nonzero constant times a combination of lifts of φ along the definite chain, restricted to Y_m. This chapter states the identity and deduces it from four theorems, proved in Chapters 44–47. We first study an analogue for one lift between lattices of ranks one and two.

43.1. A Poincaré map between definite lattices

We use the finite Poincaré map of Chapter 36, Section 36.3. Let $X := \mathcal{O}e_1$ and $Y := I_2 = X \perp \mathcal{O}e_2$ (in Section 23.4 the line X is called Y). By Problem 21.1.4 a form of weight (a) on I_n, $n \le 2$, is a single vector of $\mathrm{Sym}^a(\mathbb{C}^n)$ fixed by Γ_{I_n}, which we identify with the form; the inner product of forms is $\langle \psi, \psi' \rangle = |\Gamma_{I_n}|^{-1} \langle \psi, \psi' \rangle_{\mathrm{Sym}}$, with $\langle\,,\rangle_{\mathrm{Sym}}$ the inner product of $\mathrm{Sym}^a(\mathbb{C}^n)$. Let $0 \le j \le a$ be integers divisible by 4, and let $\varphi_j \in \mathcal{A}_{(j)}(1)$ be the form with value e_1^j. For $\varphi \in \mathcal{A}_{(j)}(1)$ let

$$P\varphi := \sum_{\gamma \in \Gamma_Y / \Gamma_{e_2}} \gamma \left(\varphi \, e_2^{a-j} \right) \in \mathcal{A}_{(a)}(2),$$

where Γ_{e_2} is the stabilizer of e_2; it is the map P of Chapter 36 with $\iota(\varphi) = \varphi \, e_2^{a-j}$. By Problem 36.2.1, $\langle \psi, P\varphi \rangle$ is the period $|\Gamma_X|^{-1} \langle \psi, \varphi \, e_2^{a-j} \rangle_{\mathrm{Sym}}$ of $\psi \in \mathcal{A}_{(a)}(2)$ along X, that of Section 23.4; and by Problem 36.3.2, $P\varphi_j = 4\big(e_1^j e_2^{a-j} + e_1^{a-j} e_2^j\big)$.

43.2. One lift

Now lift $P\varphi$ from Y to a lattice $L' = I_{n'}$, $n' \in \{1, 2\}$, with standard basis $e'_1, \ldots, e'_{n'}$; write $\mathcal{A}_{(d)}(L') := \mathcal{A}_{(d)}(n')$. For $x \colon Y \to L'$ write $x_k := x e_k$, and also view x_1 as the map $\mathbb{C}e_1 \to L'_{\mathbb{C}}$, $e_1 \mapsto x_1$. Put

$$\beta_{L'} := \sum_{v \in L'} v^{a-j} \, e^{-\pi h(v,v)} \in \mathrm{Sym}^{a-j}(L'_{\mathbb{C}}),$$

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the vector obtained from the line $\mathcal{O}e_2$ alone. Products are taken in the symmetric algebra, as in Chapter 21.

Problem 43.2.1. For $x = (x_1, x_2)$ and $\varphi \in \mathrm{Sym}^j(\mathbb{C}e_1)$, $\mathrm{Sym}^a(x)(\varphi\, e_2^{a-j}) = \big(\mathrm{Sym}^j(x_1)\varphi\big)\, x_2^{a-j}$.

Problem 43.2.2. For $\varphi \in \mathcal{A}_{(j)}(1)$, with $\Theta_{(a)}\colon \mathcal{A}_{(a)}(2) \to \mathcal{A}_{(a)}(L')$ and $\Theta_{(j)}\colon \mathcal{A}_{(j)}(1) \to \mathcal{A}_{(j)}(L')$ the lifts of Chapter 23,

$$\Theta_{(a)}(P\varphi) = \big(\Theta_{(j)}\varphi\big)\, \beta_{L'}.$$

The right side lifts φ from the smaller space X, with the kernel multiplied by the fixed vector $\beta_{L'}$.

Problem 43.2.3. $\beta_{L'}$ is fixed by $\Gamma_{L'}$. For $L' = I_1$, $\beta_{I_1} = c_{a-j}\, {e_1'}^{a-j}$ and $\Theta_{(a)}(P\varphi_j) = \frac{1}{4} c_j c_{a-j}\, {e_1'}^a$, in agreement with Problem 23.2.5. For $L' = I_2$, $a = 8$ and $j = 4$,

$$\Theta_{(8)}(P\varphi_4) = \tfrac{1}{4} c_0^2 c_4^2 \big({e_1'}^8 + 2{e_1'}^4 {e_2'}^4 + {e_2'}^8\big).$$

Problem 43.2.4. For $\psi \in \mathcal{A}_{(a)}(L')$, let $\Theta'_{(a)}\psi \in \mathcal{A}_{(a)}(2)$ be its lift back to Y. Then

$$\big\langle \Theta'_{(a)}\psi,\ P\varphi \big\rangle = \big\langle \psi,\ (\Theta_{(j)}\varphi)\, \beta_{L'} \big\rangle.$$

For $L' = I_1$ this is Problem 23.4.2.

43.3. Chains, weighted sums, and translates of the vacuum

Problem 43.3.1. In this problem take $a = 8$ and $j = 4$. Let $\Theta^{(2)}$ be the composite $\mathcal{A}_{(8)}(2) \to \mathcal{A}_{(8)}(1) \to \mathcal{A}_{(8)}(2)$ of two lifts. Then $\Theta^{(2)} P\varphi_4 = \Theta_{(8)}\big((\Theta_{(4)}\varphi_4)\, \beta_{I_1}\big) = \frac{1}{16} c_4^2 \big(c_0 c_8\, {e_1'}^8 + 70\, c_4^2\, {e_1'}^4 {e_2'}^4 + c_0 c_8\, {e_2'}^8\big)$.

By Problem 23.5.1, the kernel of $\Theta^{(2)}$ is the sum over the middle classes of the products of the two weight-eight kernels, each divided by the order of the middle stabilizer. In general the data of a chain of lifts form a tensor, with one factor for each lift, and the seesaw produces a particular tensor.

The general proof pairs both sides with a form on Y_m and transfers the Bessel period along the lower chain one lift at a time (Chapter 44). The finite variables in the resulting outer integrals are summed over finite sets of cosets of a level, the cuts, with reciprocal multiplicities as weights (Chapter 46). Here is a finite model for those weights.

Problem 43.3.2. Let a finite group G_0 act on a finite set $\mathcal{X}$, let $\mathrm{Cut} \subset \mathcal{X}$ meet every orbit, and put $\mathrm{mult}(y) := \#\{\gamma \in G_0 : \gamma^{-1} y \in \mathrm{Cut}\}$. Then for every function $f\colon \mathcal{X} \to \mathbb{C}$,

$$\sum_{y \in \mathcal{X}} f(y) = \sum_{\gamma \in G_0} \sum_{c \in \mathrm{Cut}} \frac{f(\gamma c)}{\mathrm{mult}(\gamma c)}.$$

If f is G_0-invariant, the right side is $|G_0| \sum_{c \in \mathrm{Cut}} f(c)/\mathrm{mult}(c)$; if Cut has exactly one point in each orbit, then $\mathrm{mult}(c) = |\mathrm{Stab}_{G_0}(c)|$ for $c \in \mathrm{Cut}$. For $G_0 = (\mathbb{Z}/5)^\times$ acting on $\mathcal{X} = \mathbb{Z}/5$ by multiplication, $\mathrm{Cut} = \{0, 1, 2\}$ and $f = 1$, the multiplicities are $4, 2, 2$ and both sides equal 5.

In the proof the sums are formed for a tame character ξ (Definition 46.3.2), and each multiplicity that occurs divides a power of $\operatorname{Nm} Q_\xi$ (Problem 46.3.4).

The data for the definite chain in the seesaw are finite sums of tensors built from translates of the integral vacuum of each lift. In the simplest local model the vacuum is the characteristic function $\mathbf{1}_{\Lambda_0}$ of a lattice $\Lambda_0 \subset \mathbb{Q}_p^n$, and the elementary translation operators are the shifts $f(v) \mapsto f(v-x)$ and the multiplications $f(v) \mapsto \chi(v)f(v)$ by additive characters χ of $\mathbb{Q}_p^n$.

Problem 43.3.3. Let $\Lambda \subset \Lambda_0 \subset \mathbb{Q}_p^n$ be lattices with $[\Lambda_0 : \Lambda] = p^e$. Then $\mathbf{1}_{\Lambda_0} = \sum_{x \in \Lambda_0/\Lambda} \mathbf{1}_{x+\Lambda}$, and $\mathbf{1}_\Lambda = p^{-e} \sum_\chi \chi\, \mathbf{1}_{\Lambda_0}$, where χ runs over one extension to $\mathbb{Q}_p^n$ of each of the p^e characters of Λ_0/Λ. Hence the characteristic function of every lattice commensurable with Λ_0 is a combination of shifts of the products $\chi \mathbf{1}_{\Lambda_0}$ with coefficients in $\mathbb{Z}[1/p]$.

43.4. The seesaw identity

We use the field E, the levels $M = p_0 p_1$ and N, the elements π_M and π_N, the spaces X_j and Y_j, the source forms $S \supseteq S_{\mathrm{ram}}$, tame characters ξ and twists ϑ with their moduli Q, and the characters μ_j^0 of the lifts, as in Chapter 34. Put $\varrho := m/2 - 1$.

Definition 43.4.1 (the lower chain)**.** For $0 \le j \le m$ let $K_j^{Y\circ} := K^\circ(Y_j)$ be the standard lattice stabilizer of Y_j (Section 34.3) and, for a tame character ξ, $K_j^Y(\xi) \subseteq K_j^{Y\circ}$ its subgroup of elements that reduce to the identity at the places dividing π_M and at those in Q_ξ; K_M is the principal level of Y_0 (Section 34.4), so that $K_0^Y(\xi) \subseteq K_M \subseteq K_0^{Y\circ}$. Let $R_0 = \mathcal{U} \rtimes U(X_0)$ be the Bessel subgroup of Y_0 and $\psi_\mathcal{U}$ its character (Chapter 36). The Bessel function $\mathcal{W}_{\lambda,\varphi}$ of a source form φ formed with a linear form λ, and the Poincaré map $\mathcal{P}_{g,\lambda}\colon S \to H(Y_0)$ for a point g of the archimedean unitary group of Y_0, are those of Definition 36.2.1. With the Petersson product conjugate linear in the first variable, as in this chapter, the defining identity reads $\langle y, \mathcal{P}_{g,\lambda}\varphi\rangle = \int_{[R_0]} \mathcal{W}_{\lambda,\varphi}(r)\, \overline{y(r(g,1))}\, dr$. Let υ be the fixed splitting character restricted to the norm-one torus. For a tame character ξ and $1 \le j \le m$, *lift j of the lower chain*, $\Theta_{\xi,j}$, goes from forms on Y_{j-1} to forms on Y_j. Its Weil representation ω_j, of $U(Y_{j-1})(\mathbb{A}) \times U(Y_j)(\mathbb{A})$ on the data on $\operatorname{Hom}(Y_{j-1}, Y_j)$, uses on each of the two spaces, of rank n, the character $\mu_j^0 \xi^j \upsilon^{\lfloor n/2 \rfloor}$. Its datum $\phi_j(\xi)$ is the prescribed joint harmonic at infinity times the finite datum $\mathbf{1}_j(\xi)$, given at the places dividing π_M and at those in Q_ξ by the indicator of the reduction of the inclusion matrix and at the other finite places by the vacuum. The lift is the theta lift with this datum multiplied by the index $[K_{j-1}^{Y\circ} : K_{j-1}^Y(\xi)]$. The composite $\Theta_{\xi,m} \cdots \Theta_{\xi,1}$, followed by the passage to algebraic forms on Y_m, is the lower chain Θ_ξ of Chapter 34.

Definition 43.4.2 (the definite chain)**.** Let $\mathcal{T}$ be the set of tuples $\mathfrak{t} = (\mathfrak{t}_j)_{j \ge 0}$ of matrices $\mathfrak{t}_j$ of finite adeles of E, with rows indexed by the basis of X_{j+1} and columns by that of X_j. Write $\mathrm{Heis}(\cdot)$ for the finite Heisenberg operators on the finite data on $\operatorname{Hom}(X_j, X_{j+1})$. The *translated vacuum* $v_j(\mathfrak{t})$ is the integral vacuum of $\operatorname{Hom}(X_j, X_{j+1})$ (Section 34.8) translated by $\mathrm{Heis}(\mathfrak{t}_j)$. For a tame character ξ, $1 \le j \le m$ and a finite datum v on $\operatorname{Hom}(X_{j-1}, X_j)$, $\Theta_{\xi,j}^X(v)$ is lift j of the definite chain: the theta lift from the algebraic forms on X_{j-1} to those on X_j, of the weights of Definition 34.2.1, formed with the twist ξ, the prescribed harmonic at infinity and the finite datum v; it is linear in v, and lift j uses data on $\operatorname{Hom}(X_{j-1}, X_j)$, such

as $v_{j-1}(\mathfrak{t})$. For $j \geq 0$, ω_j^X is the finite Weil action of $U(X_j)(\mathbb{A}_f) \times U(X_{j+1})(\mathbb{A}_f)$ on these data for the twist ξ (written ω^X when the lift is clear), and K_j^X is the level of X_j: the integral finite adelic unitary matrices that reduce to the identity at every $w \in \mathcal{M}$ and have last row congruent to $(0, \dots, 0, 1)$ modulo π_N; K_0^X is the level of the source forms. A *pair tensor* for pair i ($0 \leq i \leq \varrho$) is a tensor of finite data on $\mathrm{Hom}(X_{2i}, X_{2i+1})$ and on $\mathrm{Hom}(X_{2i+1}, X_{2i+2})$. For pair tensors $\Phi_0, \dots, \Phi_\varrho$,

$$\Theta_\xi^X(\Phi_\varrho \otimes \cdots \otimes \Phi_0)$$

is the composite of the $\varrho + 1$ double lifts from S to forms on X_m, pair i acting through $v \otimes v' \mapsto \Theta^X_{\xi,2i+2}(v') \circ \Theta^X_{\xi,2i+1}(v)$, extended linearly; in a tensor over pairs, later pairs stand on the left.

Definition 43.4.3 (gluing, restriction, the composite lift)**.** For $0 \leq i \leq \varrho$ the *pair gluing* Gl_i, for the double lift $X_{2i} \to X_{2i+1} \to X_{2i+2}$, sums over the vectors y of residues modulo N, each represented in $\{0, \dots, N-1\}$, indexed by a basis of X_{2i+1} and the places $w \mid \pi_N$: it translates the first factor by the last column $-y/N$ supported at these places, and the second by the conjugate last row supported at their conjugates. The *glued vacuum tensor* of $\mathfrak{t} \in \mathcal{T}$ is $\mathrm{gl}(\mathfrak{t}) := \mathrm{gl}_{\varrho+1}(\mathfrak{t})$, where

$$\mathrm{gl}_0(\mathfrak{t}) := 1 \in \mathbb{C}, \qquad \mathrm{gl}_{i+1}(\mathfrak{t}) := \mathrm{Gl}_i\big(v_{2i}(\mathfrak{t}) \otimes v_{2i+1}(\mathfrak{t})\big) \otimes \mathrm{gl}_i(\mathfrak{t}) \quad (0 \leq i \leq \varrho),$$

and for a finitely supported $\Psi \colon \mathcal{T} \to \mathbb{C}$ the glued tensor is $\Psi^{\mathrm{gl}} := \sum_{\mathfrak{t}} \Psi(\mathfrak{t})\,\mathrm{gl}(\mathfrak{t})$. The *restriction* res takes an algebraic form on X_m to one on Y_m by evaluating at $\mathrm{diag}(h, 1)$, applying a fixed branching map, and multiplying by $b_N := \prod_\tau \Phi_\tau(\bar\pi_N)^{m\Delta/2 - a_\tau}$, where Φ_τ is the embedding of the CM type at τ, and $a_\tau = a$ for $\tau \in \Phi_a$, $a_\tau = 0$ otherwise; the same letter is used for the corresponding map of automorphic forms. The *composite lift* is

$$B_{\vartheta,\Psi}(\varphi) := \mathrm{res}\,\Theta_\vartheta^X\big(\Psi^{\mathrm{gl}}\big)\varphi \qquad (\varphi \in S).$$

Definition 43.4.4 (chain data)**.** A finitely supported $\Psi \colon \mathcal{T} \to \mathbb{C}$ is a *chain datum* for the twist ϑ if: (i) for $\mathfrak{t}$ in the support of Ψ, every entry of every $\mathfrak{t}_j$ becomes integral, after multiplication by $\bar\pi_M$, at every finite place w not above a prime dividing $\mathrm{Nm}\,Q_\vartheta$, and vanishes at every w not above a prime dividing $M\,\mathrm{Nm}\,Q_\vartheta$; (ii) every value $\Psi(\mathfrak{t})$ lies in the ring $\mathcal{O}^{[\mathrm{Nm}\,Q_\vartheta]}$ of numbers integral away from $\mathrm{Nm}\,Q_\vartheta$ (Chapter 34); (iii) if $Q_\vartheta = \emptyset$, then $\sum_{\mathfrak{t}} |\sigma(\Psi(\mathfrak{t}))| \leq M^{3fmn_m^2}$ for every field automorphism σ of $\mathbb{C}$; (iv) Ψ^{gl} is fixed by $\bigotimes_j \omega_j^X(b_j, b_{j+1})$, formed for the twist ϑ, for all $b_j \in K_j^X$ ($j < m$) and $b_m = 1$.

Theorem 43.4.1 (the seesaw identity)**.** *Let D be admissible data, λ a linear form on the values of the source forms, and g a point of the archimedean unitary group of Y_0. There is $c \in \mathbb{C}^\times$ such that for every twist ϑ there is a finitely supported $\Psi \colon \mathcal{T} \to \mathbb{C}$ which is a chain datum for ϑ and satisfies*

$$\Theta_\vartheta\big(\mathcal{P}_{g,\lambda}\varphi\big) = c\,B_{\vartheta,\Psi}(\varphi) \qquad \textit{for every } \varphi \in S_{\mathrm{ram}}.$$

For fixed D, λ, g, one scalar c works for all twists, and for each twist one Ψ works for all $\varphi \in S_{\mathrm{ram}}$; the equality is one of algebraic forms on Y_m.

Chapters 44–47 prove Theorem 43.4.1, along the lines of Section 43.2. Both sides are paired with a form on Y_m, and the lifts of the lower chain are moved onto that form by transposition, as in Problem 43.2.4; Theorem 44.4.1 then carries the remaining Bessel period through one lift of the lower chain. Theorem 45.5.1 takes the transfers two at a time: after each pair, the function left on the definite group

is a coefficient of the definite chain with evaluated pair data. Theorem 46.4.1 sums the finite variables over the cuts, with weights as in Problem 43.3.2: there is $c' \in \mathbb{C}$, possibly zero, such that for every tame character ξ and every $\varphi \in S_{\mathrm{ram}}$

$$\Theta_{\xi,m} \cdots \Theta_{\xi,1} \mathcal{P}_{g,\lambda} \varphi = c' \operatorname{res} \Theta_{\xi}^{X}(\mathrm{ss}_{\xi}) \varphi,$$

an equality of automorphic forms on Y_m, with the seesaw tensor ss_{ξ} of Definition 46.4.3. Theorem 47.5.1 writes ss_{ξ} as a scalar c_{gl}, independent of ξ, times a sum of glued vacuum tensors $\mathrm{gl}(\mathfrak{t})$ whose coefficients become integral away from $\operatorname{Nm} Q_{\xi}$ after multiplication by $c_{\mathrm{res}}^{2m} = M^{2fmn_m^2}$. Passing to algebraic forms on Y_m, and applying the choice of u in Problem 43.5.4 to $c' c_{\mathrm{gl}} / c_{\mathrm{res}}^{2m}$, gives for each twist a chain datum (Problem 47.6.3) and the theorem's scalar $c = u^{-1}$.
In the Lean formalization. Theorem 43.4.1 is `Data.seesaw_identity` in the Lean proof [GL]; the objects of Definitions 43.4.1–43.4.4 are `Data.liftY`, `Data.lowerChain`, `Data.levelX`, `Data.Translations`, `Data.gluedTensor`, `Data.restrictToY`, `Data.compositeLift` and `Data.IsChainDatum`. The passage to algebraic forms is `lowerChain_eq_of_lowerChainUpTo_eq`, and the image formula for S_{ram} is `ramifiedSourceForms_eq_range`. The declarations of the four steps are named in Chapters 44–47.

43.5. Ramified forms, and what is not claimed

Where does the proof use the hypothesis that φ is ramified?

Problem 43.5.1. Let $n \geq 1$ be an integer and let F be a linear endomorphism of a finite-dimensional complex vector space V with $F^n = 1$. Then $\ker\bigl(\sum_{i=0}^{n-1} F^i\bigr) = \operatorname{im}(1 - F)$, the sum of the eigenspaces of F for eigenvalues $\neq 1$. For the cyclic shift on $\mathbb{C}^n$ this space has dimension $n - 1$.

Problem 43.5.2. Let $n \geq 1$ be an integer, and let $F_1, \ldots, F_l$ be commuting linear endomorphisms of a finite-dimensional complex vector space V with $F_k^n = 1$ for every k. Then $\bigcap_k \ker\bigl(\sum_{i=0}^{n-1} F_k^i\bigr) = \operatorname{im} \prod_k (1 - F_k)$.

So S_{ram} is the image of $\prod_{w|\pi_N} (1 - T_w)$.

Problem 43.5.3. In the situation of Problem 43.5.2, let $\Lambda \colon V \to V'$ be linear with $\Lambda F_k = \Lambda$ for one k. Then Λ vanishes on $\bigcap_k \ker\bigl(\sum_{i=0}^{n-1} F_k^i\bigr)$.

The proof uses Problem 43.5.3 for chains of lifts: if the Weil action of $(1, z_w)$ on one lift factor alone fixes the tensor of data of a chain, the chain annihilates S_{ram}. With the initial finite point in K_M, this vanishing gives the lattice bounds of Problem 46.2.16 at every stage $1 \leq j < m$; the finite Bessel orbit of a point where the stage function is nonzero therefore meets the cut of that stage (Definition 46.3.2).

Theorem 43.4.1 asserts that $c \neq 0$, but not that either side is nonzero, for any φ or any twist.

Problem 43.5.4. Let V, V' be complex vector spaces, $A, A' \colon V \to V'$ linear maps, and $c' \in \mathbb{C}$ such that $A = c' A'$. Put $u := c'^{-1}$ if $c' \neq 0$ and $u := 1$ otherwise. Then $u \neq 0$ and $A = u^{-1}(u c' A')$, and $u c' A' = 0$ if and only if $c' = 0$ or $A' = 0$.

Problem 43.5.5. For $\lambda = 0$, Theorem 43.4.1 holds with $c = 1$ and $\Psi = 0$ for every twist: $\mathcal{P}_{g,0} = 0$, the function $\Psi = 0$ is a chain datum, and $B_{\vartheta,0} = 0$.

By Theorem 36.4.2 some A_s is nonzero on S_{ram}, so $\mathcal{P}$ is not zero there; Theorem B gives no nonvanishing after the lower chain.

Hints

§43.2.

43.2.1: $e_1^j e_2^{a-j}$ is the symmetrization of $e_1^{\otimes j} \otimes e_2^{\otimes(a-j)}$, and $\mathrm{Sym}^a(x)$ is the restriction of $x^{\otimes a}$.

43.2.2: Problems 23.1.4 and 43.2.1; $h(x,x) = h(x_1,x_1)+h(x_2,x_2)$, and $\Gamma_{e_2} = \{\mathrm{diag}(u,1) : u \in \mu_4\}$ has the order of Γ_X.

43.2.3: Substitute $v \mapsto \gamma v$. For the values use Problems 23.2.3, 23.2.4 and 23.2.1, and $c_k = 0$ for $4 \nmid k$.

43.2.4: Problems 23.1.5 and 43.2.2.

§43.3.

43.3.1: Apply the second lift to both sides of Problem 43.2.2; then Problems 43.2.3 and 23.2.4.

43.3.2: Exchange the two sums on the right and group by $y = \gamma c$.

43.3.3: Orthogonality of the characters of the finite group Λ_0/Λ. Characters of Λ_0 extend to $\mathbb{Q}_p^n$. For a general lattice Λ', use $\Lambda_0 \cap \Lambda'$.

§43.5.

43.5.1: F is diagonalizable; for Problem 43.5.2 use simultaneous eigenspaces.

43.5.5: The composite lift is linear in the glued tensor.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.