A TWIST FOR EACH PRIME

Prerequisites. Finite-dimensional complex inner product spaces; Hadamard's inequality and Fischer's inequality for positive semidefinite hermitian matrices [HJ, §7.8]. From Chapter 5, *Size against divisibility*: Sections 5.3 and 5.4, whose notation we keep: isomorphisms $\iota \mid \ell$, $|\cdot|_\iota$, $\mathcal{O}_\iota$, $\bar{\iota}$, families and their heights, lattice data, index and covolume. Inner products are conjugate-linear in the first variable. For vectors $w_1, \dots, w_n$ in an inner product space let $\Gamma(w) := \det(\langle w_i, w_k\rangle)_{i,k}$, with $\Gamma(\varnothing) = 1$.

Theorem 5.4.3 compares the size and the divisibility of one Gram determinant, both read through one map Θ. In the proof the size of the Gram determinant of a basis is read through a fixed map Θ_1, and at each isomorphism $\iota \mid \ell$ its divisibility is read through the map Θ_{θ_ι} for a *twist* θ_ι allowed at ℓ. For injective maps the Gram determinant of a basis under Θ_1 is its Gram determinant under Θ_{θ_ι} times a number $Z(1)/Z(\theta_\iota)$ that does not depend on the basis. This chapter works this out on $\mathbb{C}^2$ and then states the general inequality.

6.1. One twist on $\mathbb{C}^2$

A lattice datum asks for $|\langle\varphi, \varphi'\rangle|_\iota \le 1$ with φ integral at $\bar{\iota}$ and φ' integral at ι. The two isomorphisms cannot be taken equal.

Problem 6.1.1. Some $\iota \mid 5$ has $|2 + \mathrm{i}|_\iota = 1$. For such ι, $v := (2 + \mathrm{i})^{-1}$ lies in $\mathcal{O}_\iota$, and $\langle v, v\rangle = 1/5$ does not.

A *twisted lattice* on a complex vector space V, with twists in a set Tw that has a distinguished element 1, consists of the following data.

(1) For each $\vartheta \in \mathrm{Tw}$, a complex inner product space W_ϑ and a linear map $\Theta_\vartheta \colon V \to W_\vartheta$.
(2) A predicate on $\mathbb{N} \times \mathrm{Tw}$. At a prime ℓ it specifies the twists *allowed* at ℓ. In the toys take it to be false at nonprimes.
(3) For each $\vartheta \in \mathrm{Tw}$ and each $\iota \mid \ell$, an $\mathcal{O}_\iota$-submodule $\Lambda_{\vartheta,\iota} \subseteq W_\vartheta$, such that $|\langle\varphi, \varphi'\rangle|_\iota \le 1$ for all $\varphi \in \Lambda_{\vartheta,\bar{\iota}}$ and $\varphi' \in \Lambda_{\vartheta,\iota}$ whenever ϑ is allowed at ℓ.
(4) Rational coordinates on W_1 alone: a finite set I, a linear map $b \colon W_1 \to \mathbb{C}^I$ and a matrix $G \in M_I(\mathbb{Q})$, positive definite as a complex matrix, with $\langle\varphi, \varphi'\rangle = b(\varphi)^* G\, b(\varphi')$ for all $\varphi, \varphi' \in W_1$.

A vector $y \in V$ is *integral at* $\iota \mid \ell$ if $\Theta_\vartheta y \in \Lambda_{\vartheta,\iota}$ for every ϑ allowed at ℓ; these vectors form an $\mathcal{O}_\iota$-submodule $M_\iota \subseteq V$. Sizes will be read through Θ_1 alone.

Throughout, p and r are distinct primes. The *toy* is the following twisted lattice on $V = \mathbb{C}^2$ with $\mathrm{Tw} = \{1, \vartheta\}$: $W_1 = W_\vartheta = \mathbb{C}^2$ with the standard inner product, $\Theta_1 = \mathrm{id}$, $\Theta_\vartheta = \mathrm{diag}(p^{-1}, 1)$; the twist 1 is allowed at every prime and ϑ at every prime except r; $\Lambda_{1,\iota} = \Lambda_{\vartheta,\iota} = \mathcal{O}_\iota^2$ for every ι; $I = \{1, 2\}$, $b = \mathrm{id}$, $G = 1$. Let e_1, e_2 be the standard basis.

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Problem 6.1.2. The toy is a twisted lattice. Its integral vectors are $M_\iota = p\mathcal{O}_\iota \oplus \mathcal{O}_\iota$ for $\iota \mid p$, and $M_\iota = \mathcal{O}_\iota^2$ for $\iota \mid \ell$, $\ell \neq p$.

For a linear map $u\colon U \to W$ between finite-dimensional inner product spaces, u^* is its adjoint, and $u \circ w$ is the family $u(w_1), u(w_2), \ldots$.

Problem 6.1.3. If $w_1, \ldots, w_n \in U$ and $n = \dim U$, then $\Gamma(u \circ w) = \det(u^* u)\, \Gamma(w)$.

The hypothesis $n = \dim U$ cannot be dropped.

Problem 6.1.4. In the toy, $\Gamma(\Theta_\vartheta e_1) = p^{-2}\, \Gamma(\Theta_1 e_1)$ and $\Gamma(\Theta_\vartheta e_2) = \Gamma(\Theta_1 e_2)$.

Let $Z(\vartheta) := \det(\Theta_\vartheta^* \Theta_\vartheta)$; in the toy $Z(1) = 1$ and $Z(\vartheta) = p^{-2}$. Write $\Gamma_\vartheta(w) := \Gamma(\Theta_\vartheta \circ w)$.

Problem 6.1.5. In the toy, let w_1, w_2 be a basis of $\mathbb{C}^2$ with $w_1, w_2 \in M_\iota$ for every $\iota \mid p$. Then $\Gamma_1(w) = p^2\, \Gamma_\vartheta(w)$, and $|\Gamma_1(w)|_\iota \leq p^{-2}$ for every $\iota \mid p$.

Problem 6.1.5 gives divisibility by p^2 at every $\iota \mid p$. Section 6.3 compares the gains from different choices of twist.

6.2. Index, covolume and the inequality

Fix a twisted lattice on V with twists Tw. Heights of families are those of Section 5.3. We use two facts from there: Problem 5.3.2, that the product of families of heights at most a and a' has height at most $a + a'$, and the m-th power of a family of height at most a has height at most ma for every integer $m \geq 0$; and the product formula for families, Theorem 5.3.1: if $z \in \mathbb{C}$, $z \neq 0$, the constant family z has height at most c, and $|\sigma z| \leq e^{c'}$ for every $\sigma \in \mathrm{Aut}(\mathbb{C})$, then

$$-c \leq c'.$$

Index and covolume are defined as in Section 5.4, with two changes: M_ι is now the module of vectors integral at ι for every allowed twist, and sizes are read through Θ_1, with the coordinates b and the matrix G of W_1. Thus $v_\sigma(y)$ is $b(\Theta_1 y)$ with σ applied to each coordinate, $N(y, \sigma) = v_\sigma(y)^* G\, v_\sigma(y)$, a real number ≥ 0 because G is positive definite as a complex matrix, so that the real part in Chapter 5 may be omitted, and y has covolume at most c if $|\det(v_\sigma(y_i)^* G\, v_\sigma(y_k))_{i,k}| \leq e^c$ for every σ. A family y has index at most δ if there is a function $e\colon \mathbb{N} \to \mathbb{R}$, nonnegative everywhere, with $\sum_{\ell \in U} e(\ell) \log \ell \leq \delta$ for every finite set U of primes, such that for every $\iota \mid \ell$ some $u \in M_n(\mathcal{O}_\iota)$ *carries y into M_ι with loss $e(\ell)$*: $\sum_k u_{ik} y_k \in M_\iota$ for every i, and $|\det u|_\iota \geq \ell^{-e(\ell)}$. The matrix may depend on ι but not on the twist. For the untwisted coordinates, the conjugate bound of Problem 5.4.8 holds for every twisted lattice, with the same proof: the modules $\Lambda_{\vartheta,\iota}$ play no role. If y has covolume at most c, then

$$|\sigma(\Gamma_1(y))| \leq e^c \quad \text{for every } \sigma \in \mathrm{Aut}(\mathbb{C}).$$

Let $y_1, \ldots, y_n$ and $z_1, \ldots, z_{n'}$ be vectors in V. In Problems 6.2.1–6.2.3, assume that V is a complex inner product space. Write $y \sqcup z$ for the family $y_1, \ldots, y_n, z_1, \ldots, z_{n'}$, and $|\gamma|^2$ for the complex number $\bar{\gamma}\gamma$.

Problem 6.2.1. Let ϑ be allowed at ℓ, let $\iota \mid \ell$, let $w_1, \ldots, w_m \in V$, and let $u, u' \in M_m(\mathbb{C})$ be such that $\sum_k u_{ik} \Theta_\vartheta w_k \in \Lambda_{\vartheta,\iota}$ and $\sum_k u'_{ik} \Theta_\vartheta w_k \in \Lambda_{\vartheta,\bar{\iota}}$ for every i. Then

$$|\det u'|_{\bar{\iota}}\, |\Gamma_\vartheta(w)|_\iota\, |\det u|_\iota \leq 1.$$

Problem 6.2.2. Let ϑ be allowed at ℓ, let $\iota \mid \ell$ and $a \in \mathbb{R}$. Suppose that $u \in M_n(\mathcal{O}_\iota)$ carries y into M_ι with loss a, that $u' \in M_n(\mathcal{O}_{\bar\iota})$ carries y into $M_{\bar\iota}$ with loss a, and that $\gamma \in \mathbb{C}$ satisfies $\gamma\,\Theta_\vartheta z_i \in \Lambda_{\vartheta,\iota} \cap \Lambda_{\vartheta,\bar\iota}$ for every i. Then

$$\left|(|\gamma|^2)^{n'}\,\Gamma_\vartheta(y \sqcup z)\right|_\iota \le \ell^{2a}.$$

Problem 6.2.3. Let y have index at most δ. For each $\iota \mid \ell$ let $\theta_\iota \in \mathrm{Tw}$ be allowed at ℓ and $\gamma_\iota \in \mathbb{C}$, with $\gamma_\iota\,\Theta_{\theta_\iota} z_i \in \Lambda_{\theta_\iota,\iota} \cap \Lambda_{\theta_\iota,\bar\iota}$ for every i. Then the family $\iota \mapsto (|\gamma_\iota|^2)^{n'}\,\Gamma_{\theta_\iota}(y \sqcup z)$ has height at most 2δ.

Problem 6.2.4. If y has covolume at most c, and $N(z_i, \sigma) \le e^h$ for every i and every $\sigma \in \mathrm{Aut}(\mathbb{C})$, then $y \sqcup z$ has covolume at most $c + n'h$.

Now let V be a finite-dimensional complex inner product space. For $\vartheta \in \mathrm{Tw}$ let $Z(\vartheta)$ be the square of the absolute value of the determinant of Θ_ϑ, as a map $V \to \Theta_\vartheta(V)$, in orthonormal bases, when Θ_ϑ is injective, and $Z(\vartheta) := 0$ otherwise. When W_ϑ is finite-dimensional, $Z(\vartheta) = \det(\Theta_\vartheta^*\Theta_\vartheta)$, as in Section 6.1.

Problem 6.2.5. Let all Θ_ϑ be injective, let $y \sqcup z$ be a basis of V, and let $\gamma \in \mathbb{C}$, $\gamma \neq 0$. Then $\Gamma_1(y \sqcup z) \neq 0$, and for every $\vartheta \in \mathrm{Tw}$

$$\Gamma_1(y \sqcup z) = (|\gamma|^2)^{n'}\,\Gamma_\vartheta(y \sqcup z) \cdot (|\gamma|^{-2})^{n'} \cdot \frac{Z(1)}{Z(\vartheta)}.$$

Problems 6.2.3, 6.2.4 and 6.2.5, together with the height bounds and the product formula of Chapter 5, give the following theorem.

Theorem 6.2.1. *Let V be a finite-dimensional complex inner product space,* Tw *a set with a distinguished element* 1*, and L a twisted lattice on V with twists* Tw *whose maps Θ_ϑ are all injective. Let $y_1, \dots, y_n, z_1, \dots, z_{n'} \in V$ be linearly independent, with $\dim V = n + n'$. For each $\iota \mid \ell$, for every prime ℓ, let $\theta_\iota \in \mathrm{Tw}$ be allowed at ℓ. Let $c, \delta, c_{\mathrm{vol}}, c_{\mathrm{short}}, c_{\mathrm{den}} \in \mathbb{R}$, and suppose that y has covolume at most c and index at most δ. Let $\gamma\colon \mathrm{Tw} \to \mathbb{C}$ with $\gamma(\vartheta) \neq 0$ for every ϑ, and suppose that*

1. *$\gamma(\vartheta)\,\Theta_\vartheta z_i \in \Lambda_{\vartheta,\iota}$ for every i, every $\iota \mid \ell$ for every prime ℓ, and every ϑ allowed at ℓ;*
2. *$N(z_i, \sigma) \le e^{c_{\mathrm{short}}}$ for every i and every $\sigma \in \mathrm{Aut}(\mathbb{C})$;*
3. *the family $\iota \mapsto (|\gamma(\theta_\iota)|^2)^{-1}$ has height at most c_{den};*
4. *the family $\iota \mapsto Z(1)/Z(\theta_\iota)$ has height at most c_{vol}.*

Then

$$0 \le c + 2\delta + n'(c_{\mathrm{short}} + c_{\mathrm{den}}) + c_{\mathrm{vol}}.$$

Hypothesis (1) concerns every twist allowed at ℓ, including θ_ι when it differs from $\theta_{\bar\iota}$. To recover Theorem 5.4.3, give V the inner product in which its given basis y is orthonormal, and take $\mathrm{Tw} = \{1\}$, with 1 allowed at every prime, $n' = 0$, $\gamma = 1$ and $c_{\mathrm{short}} = c_{\mathrm{den}} = c_{\mathrm{vol}} = 0$.

In the Lean formalization. In the Lean proof [GL] (file `grh/src/New/Reduction/Lattices/TwistedLattice.lean`), a twisted lattice is `New.TwistedLattice`; the integral vectors, carrying with loss, index, covolume, $N(y, \sigma)$ and $Z(\vartheta)$ are `integralVectors`, `New.PrimePadicIso.Carries`, `IndexLe`, `CovolumeLe`, `normSqAt` and `volume`. Admissibility is tested only at primes. The function e is nonnegative on all of $\mathbb{N}$, and only its prime values occur in the sums and the carrying bounds. Theorem 6.2.1 is `New.TwistedLattice.covolume_add_index_nonneg`, whose `b_volume`, `b_short`, `b_denom` are $c_{\mathrm{vol}}, c_{\mathrm{short}}, c_{\mathrm{den}}$.

6.3. A twist for each prime

We first test Theorem 6.2.1 on the toy, then add a twist that is allowed at r.

Problem 6.3.1. In the toy, let $k \in \mathbb{Z}$ and $s \geq 1$ an integer, and take $n = n' = 1$, $y_1 = p^k e_1$, $z_1 = s^{-1} e_2$, $\gamma(1) = \gamma(\vartheta) = s$, $\theta_\iota = \vartheta$ for $\iota \mid \ell$ with $\ell \neq r$, and $\theta_\iota = 1$ for $\iota \mid r$. The hypotheses of Theorem 6.2.1 hold with

$$c = 2k \log p, \quad \delta = \max(1-k, 0) \log p, \quad c_{\mathrm{short}} = -2 \log s, \quad c_{\mathrm{den}} = 2 \log s, \quad c_{\mathrm{vol}} = -2 \log p,$$

and then $c + 2\delta + n'(c_{\mathrm{short}} + c_{\mathrm{den}}) + c_{\mathrm{vol}} = 2 \max(k-1, 0) \log p$. For $p = 3$, $k = -1$, $s = 2$: $c = -2 \log 3$ and $\delta = 2 \log 3$. For the constant choice $\theta_\iota = 1$, with the same c_{short} and c_{den}, the theorem gives only $c + 2\delta \geq 0$.

The *second toy* adds to the toy a twist ϑ': $\mathrm{Tw} = \{1, \vartheta, \vartheta'\}$, $W_{\vartheta'} = \mathbb{C}^2$, $\Theta_{\vartheta'} = \mathrm{diag}(1, r^{-1})$, $\Lambda_{\vartheta', \iota} = \mathcal{O}_\iota^2$, and ϑ' is allowed at every prime except p.

Problem 6.3.2. The second toy is a twisted lattice, with $M_\iota = p\mathcal{O}_\iota \oplus \mathcal{O}_\iota$ for $\iota \mid p$, $M_\iota = \mathcal{O}_\iota \oplus r\mathcal{O}_\iota$ for $\iota \mid r$, and $M_\iota = \mathcal{O}_\iota^2$ otherwise. For $y = (pe_1, re_2)$, $n' = 0$, $\theta_\iota = \vartheta$ for $\iota \mid p$, $\theta_\iota = \vartheta'$ for $\iota \mid r$, and $\theta_\iota = 1$ otherwise, the hypotheses of Theorem 6.2.1 hold with $\gamma = 1$, $c_{\mathrm{short}} = c_{\mathrm{den}} = 0$, $c = 2 \log(pr)$, $\delta = 0$ and $c_{\mathrm{vol}} = -2 \log(pr)$, and the inequality is an equality.

Problem 6.3.3. In the second toy, suppose that $\theta_\iota \in \{1, \vartheta\}$ for every ι. If $\iota \mapsto Z(1)/Z(\theta_\iota)$ has height at most c'', then $c'' \geq -2 \log p$. The same holds with ϑ' and r in place of ϑ and p.

So for y as in Problem 6.3.2, a choice of twists with values in $\{1, \vartheta\}$ or in $\{1, \vartheta'\}$ yields through Theorem 6.2.1 at most $c \geq 2 \log p$ or $c \geq 2 \log r$, while the choice that depends on the prime yields $c \geq 2 \log(pr)$, which is attained.

Under the other hypotheses of Theorem 6.2.1, a height bound of $-A$ for $\iota \mapsto Z(1)/Z(\theta_\iota)$ gives $c + 2\delta \geq A - n'(c_{\mathrm{short}} + c_{\mathrm{den}})$. The proof of Theorem C (Chapter 54) contains the sole application of Theorem 6.2.1. There V is the space T of target forms, and Tw is the set of m-fold twists. A twist ϑ with modulus Q_ϑ is allowed at ℓ when ℓ is prime to $\mathrm{Nm}\, Q_\vartheta$. Its volume $Z(\vartheta)$ is the Gram determinant of the images of an orthonormal basis under the chain of lifts (Chapter 55). The height of $\iota \mapsto Z(1)/Z(\theta_\iota)$ is bounded by combining the bounds for the single lifts of Chapter 60 with a height bound for the inverse scalar in the chain factorization of Chapter 55. The vectors z_i are chosen from the auxiliary forms of Section 55.5 to complete y to a basis, and each $\gamma(\vartheta)\Theta_\vartheta z_i$ is integral at every ι at which ϑ is allowed.

Hints

6.1.1: Problem 5.3.1.
6.1.2: Problem 5.4.1 for the pairing condition. At p the twist ϑ is allowed; elsewhere p is a unit.
6.1.3: Write $\Gamma(w) = |\det P|^2$ with P the matrix of w in an orthonormal basis.
6.1.5: Problem 6.1.3; for $\Gamma_\vartheta(w)$, the pairing condition at ι and $\bar{\iota}$.
6.2.1: Form the matrix of pairings of the rows formed with u' against those formed with u, in that order.
6.2.2: Block diagonal matrices.
6.2.3: Take the multiplier 1. The index gives matrices at ι and at $\bar{\iota}$, which lie above the same prime.

6.2.4: Fischer's inequality, then Hadamard's.
6.2.5: Problem 6.1.3 with target $\Theta_\vartheta(V)$ and its induced inner product.
6.3.1: Problem 6.1.2 for the index. For c_{vol} take $H = p^{-2}$.
6.3.2: For c_{vol} take $H = (pr)^{-2}$.
6.3.3: Compare with the constant family p^2 at every ι.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[HJ] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, 2013.