THEOREM B: ENOUGH FORMS

Prerequisites. From Chapter 10: Poincaré series on $\mathrm{SL}_2(\mathbb{Z})$, the unfolding of Problem 10.1.1, Problem 10.1.3, and the main term of the dimension (Section 10.4). From Chapter 21: class sets, the groups Γ_L, forms and their norm (Section 21.3), Problems 21.1.2, 21.1.4, 21.3.2 and 21.3.5, and the mass of rank two, written here $\mathrm{mass}_2 = 1/32$. From Chapter 23: the period of Section 23.4. From Chapter 9: Hilbert–Schmidt norms, Problems 9.2.2, 9.2.5 and 9.3.1, near isometries, and Theorems 9.3.1 and 9.4.1. From Chapter 33: the data, and the spaces and forms of Chapter 34 (Y_0, X_0, the chosen places $\mathcal{M}$, the group $\bar{G}$, the idempotent e_0, the forms $H(Y_0)$, the target T of dimension d_T, the source $S \supseteq S_{\mathrm{ram}}$ of dimension D_X, and the operators T_w). The Bessel periods of Gan, Gross and Prasad [GGP, §§12, 23], at the level of the definition. Norms of linear maps are Hilbert–Schmidt norms, $\|L\|^2 = \mathrm{Tr}(L^*L)$, and log is the natural logarithm.

In Chapter 10 a Fourier coefficient of a cusp form is, up to an explicit factor, its inner product with a Poincaré series, and Petersson's formula expands the inner products of two Poincaré series. In the proof, a Poincaré map represents Bessel periods on a unitary group, and the Gram operators between it and its translates by a finite group have an expansion in integrals indexed by rational points. After projection to the good forms and restriction to the ramified source, a finite set of translates spans almost all of the target (Theorem 36.4.2). The estimates behind this are developed in Chapters 37, 38 and 41. How does the Gram operator group into double cosets in a finite example, and what does a second moment over translates give? Hermitian forms h and inner products are linear in the first variable; Chapter 9 uses the other convention, and the statements recalled from it do not depend on the choice.

36.1. The Bessel subgroup

Let E/F be a CM extension, ψ_F a nontrivial character of $F\backslash\mathbb{A}_F$, and let V be a hermitian space over E of rank r with a decomposition

$$V = Ev_1 \oplus \cdots \oplus Ev_\varrho \oplus X \oplus Ez \oplus Ev_1' \oplus \cdots \oplus Ev_\varrho',$$

where $h(v_a, v_c) = h(v_a', v_c') = 0$, $h(v_a, v_c') = \delta_{ac}$, the space X of rank k is orthogonal to the v_a, v_a' and z, and z is orthogonal to every v_a and v_a', with $h(z,z) = -N$ for a prime N. So $r = 2\varrho + k + 1$. This is the shape of Y_0 in Chapter 33, with $\varrho = m/2 - 1$, $X = X_0$, and N the prime of the data.

Let $\mathcal{U}$ be the group of $n \in U(V)$ such that $(n-1)v_a$ lies in the span of $v_1, \ldots, v_{a-1}$ for every a, and $(n-1)y$ lies in the span of $v_1, \ldots, v_\varrho$ for every y orthogonal to all v_a. It is the unipotent radical of the stabilizer of the isotropic flag $Ev_1 \subset Ev_1 + Ev_2 \subset \cdots$. Embed $U(X)$ in $U(V)$ as the pointwise stabilizer of the v_a, the v_a' and z. Put

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$v_{\varrho+1} := z$ and, for $n \in \mathcal{U}$,

$$\lambda(n) := \sum_{a=1}^{\varrho} h(nv_{a+1}, v'_a), \qquad \psi_{\mathcal{U}}(n) := \psi_F\big(\mathrm{Tr}_{E/F}\,\lambda(n)\big).$$

The *Bessel subgroup* is $R := \mathcal{U} \rtimes U(X)$. For scalar-valued automorphic forms y on $U(V)$ and ϕ on $U(X)$ for which the integral converges absolutely, the Bessel period integrates y against $\overline{\psi_{\mathcal{U}}}$ over $\mathcal{U}(F)\backslash\mathcal{U}(\mathbb{A}_F)$, and the result against $\bar{\phi}$ over $U(X)(F)\backslash U(X)(\mathbb{A}_F)$ [GGP, §§12, 23]. For $\varrho = 0$ the group $\mathcal{U}$ is trivial, and the Bessel period is the restriction period along $U(X) \subset U(X \perp Ez)$.

Problem 36.1.1. For $n \in \mathcal{U}$, $\lambda(n)$ is the sum over $a \le \varrho$ of the coefficient of v_a in nv_{a+1}. The map $\lambda\colon \mathcal{U} \to E$ is a homomorphism, and $\lambda(gng^{-1}) = \lambda(n)$ for $g \in U(X)$. Hence $\psi_{\mathcal{U}}$ is a character of $\mathcal{U}(\mathbb{A}_F)$, trivial on $\mathcal{U}(F)$ and fixed by $U(X)$.

Problem 36.1.2. $U(X)$ normalizes $\mathcal{U}$, and $U(X) \cap \mathcal{U} = \{1\}$.

Problem 36.1.3. As an algebraic group over F, $\mathcal{U}$ has dimension $\varrho(r+k) = 2\varrho(r-2\varrho) + \varrho^2 + \varrho(\varrho-1)$. For $(\varrho, k) = (1, 0)$ it is the three-dimensional unipotent radical of a Borel subgroup of $U(V)$, V of rank 3.

36.2. Poincaré series

The Poincaré series of the proof plays the role of that of Chapter 10: the Bessel subgroup replaces Γ_∞ in the period, and a Bessel function replaces $e(mz)$. Take $V = Y_0$, $X = X_0$ and $G := U(Y_0)$, with the forms $H(Y_0)$, a finite-dimensional space with the Petersson inner product, and the source forms S, as in Chapter 34. On the Bessel subgroup of the data the proof fixes a generic character $\psi_{\mathcal{U}}$ and a quotient measure; no comparison with the character of Section 36.1 is made here.

Definition 36.2.1 (Poincaré maps)**.** Let λ be a linear form on the values of the source forms and g a point of the archimedean unitary group of Y_0. (From here on λ is such a linear form; the homomorphism $\lambda\colon \mathcal{U} \to E$ of Section 36.1 enters only through $\psi_{\mathcal{U}}$.) For $\varphi \in S$, the *Bessel function of φ formed with λ* is $\mathcal{W}_{\lambda,\varphi}(nh) := \psi_{\mathcal{U}}(n)\,\lambda\big(h_\infty^{-1}\varphi(h_f)\big)$ for $n \in \mathcal{U}(\mathbb{A}_F)$ and $h \in U(X_0)(\mathbb{A}_F)$, where h_∞^{-1} acts on the values. By the Riesz representation theorem there is a unique $\mathcal{P}_{g,\lambda}\varphi \in H(Y_0)$ with

$$\langle \mathcal{P}_{g,\lambda}\varphi, y\rangle = \int_{R(F)\backslash R(\mathbb{A}_F)} \mathcal{W}_{\lambda,\varphi}(x)\,\overline{y(xg)}\,dx \qquad (y \in H(Y_0)).$$

The linear map $\mathcal{P}_{g,\lambda}\colon S \to H(Y_0)$ is the *Poincaré map* of (g, λ): it represents a Bessel period of φ, linear in φ and conjugate-linear in y. The proof fixes one pair (g, λ); $\mathcal{P}$ without subscripts is the Poincaré map of this pair.

For the fixed pair, $\mathcal{P}\varphi$ is a Poincaré series for every φ: the sum over $\gamma \in G(F)$ of the translates of a seed made of each coordinate of the archimedean Bessel function, multiplied by a cutoff at the infinite places and by the indicator of a translate of the level at the finite places, one for each class; the coefficients are the Bessel coordinates of the values of a central translate of φ at class representatives.

At a fixed level on a definite group with one class, the forms are the vectors invariant under a finite group, and the series becomes a finite sum. Let Γ be a finite group acting unitarily on a space V_π, and let the forms on G be the vectors V_π^Γ, with $\|\psi\|^2 := |\Gamma|^{-1}\|\psi\|_{V_\pi}^2$, as in Chapter 21 for one class. Let $\Gamma_X \subset \Gamma$ be a subgroup, W a unitary representation of Γ_X, and $\jmath\colon W \to V_\pi$ a Γ_X-equivariant

linear map with $\|\jmath\xi\|^2_{V_\pi} = c_0\|\xi\|^2_W$ for a constant $c_0 > 0$. The forms on X are W^{Γ_X}, with $\|\xi\|^2 := |\Gamma_X|^{-1}\|\xi\|^2_W$. For $\psi \in V_\pi^\Gamma$ and $\xi \in W^{\Gamma_X}$ put

$$\mathrm{Per}(\psi,\xi) := |\Gamma_X|^{-1}\langle\psi,\jmath\xi\rangle_{V_\pi}, \qquad P\xi := \sum_{\gamma\in\Gamma/\Gamma_X} \gamma\,\jmath\xi.$$

Problem 36.2.1. $P\xi$ is well defined and lies in V_π^Γ, and $\langle\psi, P\xi\rangle = \mathrm{Per}(\psi,\xi)$ for every $\psi \in V_\pi^\Gamma$, where the left side is the inner product of forms.

Problem 36.2.2. The adjoint of $P\colon W^{\Gamma_X} \to V_\pi^\Gamma$ for the two norms of forms is $P^*\psi = \jmath^*\psi$, where $\jmath^*$ is the adjoint for the inner products of W and V_π. In particular $P \neq 0$ if and only if $\mathrm{Per}(\psi,\xi) \neq 0$ for some ψ and ξ.

Problem 36.2.3. Let $(W',\jmath')$ be a second pair as above, with Poincaré map P'. For $\xi \in W^{\Gamma_X}$ and $\xi' \in W'^{\Gamma_X}$,

$$\langle P\xi, P'\xi'\rangle = \sum_{\delta\in\Gamma_X\backslash\Gamma/\Gamma_X} \frac{\langle\delta\,\jmath\xi, \jmath'\xi'\rangle_{V_\pi}}{|\Gamma_X\cap\delta\Gamma_X\delta^{-1}|},$$

and if $(W',\jmath') = (W,\jmath)$ the term of $\delta = 1$ is $c_0\langle\xi,\xi'\rangle$, the inner product of forms.

So $\mathcal{K} := c_0^{-1}P^*P$ is the identity plus a sum over the nontrivial double cosets. This is the finite form of Petersson's formula, Theorem 10.2.1, where the double cosets $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ give the Kloosterman terms.

36.3. The smallest case

Take $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$, $\Gamma := \Gamma_{I_2}$, of order 32 by Problem 21.1.2, and $\mathrm{Cl}_2 = \{I_2\}$ by Problem 21.1.4. Let $\mathbf{e}_1, \mathbf{e}_2$ be the standard basis of $\mathbb{C}^2$, $z := \mathbf{e}_2$ and $X := \mathcal{O}\mathbf{e}_1$; here $h(z,z) = 1$, since everything is definite. The stabilizer of z is $\Gamma_X = \{\mathrm{diag}(u,1) : u \in \mu_4\}$. In this section, as in Chapter 21, $a \geq 0$ is the degree of a symmetric power (not the scale of Chapter 33), and $0 \leq j, j' \leq a$ are integers. Let $V_\pi := \mathrm{Sym}^a(\mathbb{C}^2)$, $W := \mathrm{Sym}^j(\mathbb{C}\mathbf{e}_1)$ and $\jmath(\xi) := \xi\,\mathbf{e}_2^{a-j}$. When $4 \mid j$ and $4 \mid a-j$, Per is the period $P(\psi,\xi)$ of Section 23.4, and Problem 36.2.1 says that it is the inner product with $P\xi$. For $4 \mid j$ let $\xi_j \in W^{\Gamma_X}$ be the form with value $\mathbf{e}_1^j$.

Problem 36.3.1. $\Gamma_X\backslash\Gamma/\Gamma_X$ has five elements: the classes of $\mathrm{diag}(1,u)$, $u \in \mu_4$, with four elements each, and the class of the sixteen antidiagonal matrices.

Problem 36.3.2. Let $4 \mid j$. Then $P\xi_j = 4\big(\mathbf{e}_1^j\mathbf{e}_2^{a-j} + \mathbf{e}_1^{a-j}\mathbf{e}_2^j\big)$ if $4 \mid a$, and $P\xi_j = 0$ otherwise. For $a = 8$: $P\xi_0 = 4(\mathbf{e}_1^8 + \mathbf{e}_2^8)$ and $P\xi_4 = 8\mathbf{e}_1^4\mathbf{e}_2^4$.

Problem 36.3.3. Here $c_0 = \binom{a}{j}^{-1}$. Let $4 \mid a$ and $4 \mid j, j'$, and put $\mathcal{K}_{jj'} := \big(c_0(j)c_0(j')\big)^{-1/2}\langle P\xi_j, P\xi_{j'}\rangle/(\|\xi_j\|\|\xi_{j'}\|)$. Then

$$\mathcal{K}_{jj'} = 4\big(\delta_{jj'} + \delta_{j',a-j}\big).$$

In the expansion of Problem 36.2.3 the identity class contributes $\delta_{jj'}$ to $\mathcal{K}_{jj'}$, the three classes $\mathrm{diag}(1,u)$ with $u \neq 1$ together contribute $3\delta_{jj'}$, and the antidiagonal class contributes $4\delta_{j',a-j}$.

Problem 36.3.4. Let $4 \mid a$. The forms $P\xi_j$ with $4 \mid j$ and $2j \leq a$ are an orthogonal basis of $\mathcal{A}_{(a)}(2)$, their number is $\lfloor a/8\rfloor + 1$, and $P\xi_{a-j} = P\xi_j$.

Here different indices can give the same series; in Problem 10.1.3 two Poincaré series are proportional.

Problem 36.3.5. Let $4 \mid a$ and $D^* := 4\,\mathrm{mass}_2 \dim \mathrm{Sym}^a(\mathbb{C}^2)$. Then $D^* = (a+1)/8$, and $0 < \dim \mathcal{A}_{(a)}(2) - D^* \le 7/8$.

So for $4 \mid a$ the series $P\xi_j$ with $4 \mid j$ and $0 \le 2j < a$ are orthogonal and span a subspace of codimension at most one, and $\dim \mathcal{A}_{(a)}(2)$ is D^* up to a bounded error, as $\dim S_k$ is $(k-1)/12$ up to a bounded error for even $k \ge 4$ in Problem 10.4.2. But their diagonal Gram entries are 4, not 1: the elements $\mathrm{diag}(1,u)$ with $u \ne 1$, which preserve the line Ez without fixing z, each contribute as much as the identity. The toy has nothing to average over. (Problems 37.1.7 and 37.1.6 make this precise.) In the proof a finite level group acts on the forms of principal level and translates one Poincaré map.

36.4. Translates of one Poincaré map

The proof gives three estimates for the scaled Poincaré map of the data: a second moment over the finite group (Theorem 40.3.1), the near isometry of its projection to the good forms (Theorem 41.2.2), and an upper bound for the dimension of the target (Theorem 41.3.1). We apply the two translate theorems of Chapter 9 to obtain Theorem 36.4.1 and, using also the dimension bound for S_{ram}, Theorem 36.4.2. How small can the second moment of the Gram operators between a Poincaré map and its translates be, and how does its upper bound combine with near isometry of the projected map to give a spanning estimate?

Let ρ be a unitary representation of a finite group $\bar{G}$ on a finite-dimensional space $\mathcal{Y}$, let $\mathcal{X}$ be a finite-dimensional inner product space, and let $\mathcal{P}^\circ \colon \mathcal{X} \to \mathcal{Y}$ be linear. Put $\mathcal{K}_s := \mathcal{P}^{\circ *}\rho(s)\mathcal{P}^\circ \in \mathrm{End}\,\mathcal{X}$ for $s \in \bar{G}$; it is the *Gram operator* between $\mathcal{P}^\circ$ and its translate $\rho(s)\mathcal{P}^\circ$. Recall two facts from Chapter 9. By Problem 9.2.2, $\mathrm{Tr}\big((\rho(s)\mathcal{P}^\circ\mathcal{P}^{\circ *}\rho(s)^{-1})(\rho(t)\mathcal{P}^\circ\mathcal{P}^{\circ *}\rho(t)^{-1})\big) = \|\mathcal{K}_{s^{-1}t}\|^2$, so $s \mapsto \|\mathcal{K}_s\|^2$ is of *positive type*: there are vectors u_s, $s \in \bar{G}$, of an inner product space with $\langle u_s, u_t\rangle = \|\mathcal{K}_{s^{-1}t}\|^2$. By Problem 9.2.5, if p is the orthogonal projection onto a $\bar{G}$-stable subspace $\mathcal{Y}_0$ and $p\mathcal{P}^\circ$ is an isometry, then $\dim \mathcal{Y}_0 \sum_{s\in\bar{G}} \|\mathcal{K}_s\|^2 \ge \#\bar{G}\,(\dim \mathcal{X})^2$. Chapter 37 computes the Gram operators of a Poincaré map on a finite group: there the lower bound is attained when the weight is irreducible (Problems 37.1.4 and 37.2.1), and the translates by a subgroup act on the source through a representation, as $\bar{R}_0$ does below.

Definition 36.4.1 (the scaled Poincaré map)**.** We use $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1$, $\bar{G} = \bar{G}_0 \times \bar{G}_1$, ρ on $H(Y_0)$, e_0, $T = e_0H(Y_0)$, d_T, $S \supseteq S_{\mathrm{ram}}$ and $D_X = \dim S$ as in Chapter 34; for the data, $b \ge 1000$. A finite adelic element of $U(Y_0)$ is *integral* if its matrix in the lattice basis of the proof, to which the vectors v_a, z, v_a' of Section 36.1 belong, is integral at every finite place; its *reduction* is then the family of its residues at the places $w \in \mathcal{M}$, an element of $\bar{G}$. The integral elements with reduction 1 form the principal level K_M. The *classes* c are the double cosets of the finite adelic points of $U(X_0)$ by the rational points and the level of the source forms; each has a representative h_c, viewed in $U(Y_0)$. A *Bessel coordinate* $i = (i_\tau)_\tau$ is a tuple of coordinates of the archimedean Bessel functions, one at each infinite place. The *coordinate space* has an orthonormal basis $e_{c,i}$ indexed by the pairs of a class and a Bessel coordinate. Let C be the coefficient isomorphism of the Poincaré series of Section 36.2, from S to the coordinate space: it translates a source form by a fixed central element and evaluates it at the class representatives in Bessel coordinates. Give S the inner product transported by C. Let $\nu > 0$ be the

covolume, for the product of the local Bessel Haar measures on the real unipotent group, of the lattice of $n \in \mathcal{U}(F)$ with $h_c^{-1} n_f h_c \in K_M$; it is the same for every class. (In this Part the letter ν has no other meaning.) The *scaled Poincaré map* and its *Gram operators* are

$$\mathcal{P}^\circ := (\mathrm{vol}(K_M)\,\nu)^{-1/2}\,\mathcal{P} \circ C^{-1}, \qquad \mathcal{K}_s := \mathcal{P}^{\circ *}\rho(s)\mathcal{P}^\circ \quad (s \in \bar{G}),$$

a map from the coordinate space to $H(Y_0)$ and operators on the coordinate space. Further, G_∞ is the product over the infinite places of the real groups of Y_0, in the rescaled bases of the proof, and dy its product measure; γ_∞ and γ_f are the components of a rational $\gamma \in U(Y_0)(F)$ in G_∞ and in the finite adelic group; deg is the product over the infinite places of the degree constants of the Bessel systems (formal degrees), and mass_r is the mass $\sum 1/\#\mathrm{Aut}$ of the genus of the standard lattice of rank r.

Definition 36.4.2 (the Bessel representation)**.** Let $\bar{R}_0 \subseteq \bar{G}$ be the group of reductions of the integral points of the finite adelic Bessel subgroup. For $s \in \bar{G}$, $\rho_{\mathrm{Bes}}(s)$ is the operator on the coordinate space whose $((c,i),(c',i'))$ entry is the complex conjugate of ν^{-1} times the sum, over the $\gamma \in R(F)$ with $h_c^{-1}\gamma_f h_{c'}$ integral of reduction s, of the integrals

$$\int_{G_\infty} u_c(y)\, u_{c'}(\gamma_\infty^{-1} y)\, W_i(y)\, \overline{W_{i'}(\gamma_\infty^{-1} y)}\, dy.$$

Here u_c is the class cutoff and W the full Bessel function; both are made precise in Chapter 37 (Theorem 37.3.1 and Definition 37.3.1), in whose notation the entry is the complex conjugate of $\nu^{-1}\mathrm{Amp}(R(F); c, s, c')_{ii'}$. The proof shows that $\rho_{\mathrm{Bes}}(s) = 0$ for $s \notin \bar{R}_0$ and that ρ_{Bes} is a unitary representation of $\bar{R}_0$ on the coordinate space.

The same sums taken over all $\gamma \in U(Y_0)(F)$ give the entries of $\mathcal{K}_s$: this is the block identity, Theorem 37.3.2, the analogue of Problem 36.2.3. It implies $\rho(t)\mathcal{P}^\circ = \mathcal{P}^\circ \rho_{\mathrm{Bes}}(t)$ for $t \in \bar{R}_0$ (Problem 37.4.1).

Fillings. The proof establishes the estimates of this Part not only for the data. A *filling* consists of a CM field E with a CM type, the integers ϱ and k of Section 36.1, a prime N, two further primes p_0, p_1 with, for each, a set of f chosen places of norm p_i (f the number of infinite places of E; no chosen place is conjugate to a chosen place), a real $b \geq 1000$, a level in $U(X_0)$ whose classes have trivial stabilizers, and a Bessel system at each infinite place, subject to size conditions in powers of b. It is *admissible* if it satisfies the further standing hypotheses of the proof; the data give an admissible filling. For a filling, $\bar{G}$, the classes, the coordinate space, ν, $\bar{R}_0$, ρ_{Bes}, deg and mass_r are defined as above, D_X is the dimension of the coordinate space, and a linear map from the coordinate space to a representation ρ of $\bar{G}$ *satisfies the block identity* if the identity of Theorem 37.3.2 holds with this map in place of $\mathcal{P}^\circ$. The field has *Weil data* if it contains a primitive 5040-th root of unity and carries the further choices, a splitting character among them, with which the proof forms theta lifts; the field of the data has them. The two properties of ρ_{Bes} in Definition 36.4.2 hold for every admissible filling over a field containing a primitive 5040-th root of unity. Chapters 38, 39, 40 and 41 say for which fillings each estimate is proved; the reader may take the data throughout.

The second moment over the whole group is Theorem 40.3.1: with $D_{\text{main}} = D_X^2/(\deg \cdot \text{mass}_r)$ (Definition 37.3.2),

$$\sum_{s\in\bar G} \|\mathcal{K}_s\|^2 \le (1+9b^{-4})\, D_{\text{main}}.$$

Chapter 40 proves it from the estimates of Chapters 38 and 39, which rest on statements read from the proof. Two of them, which Chapter 38 derives, concern the reductions near the Bessel subgroup: for $j = 0, 1$, the sum of $\|\mathcal{K}_s - \rho_{\text{Bes}}(s)\|^2$ over the $s \in \bar G$ whose component in $\bar G_j$ equals that of an element of $\bar R_0$ is at most $b^{-12}\, \#\bar R_0\, D_X$ (Theorem 38.3.3); hence $\|\mathcal{P}^{\circ *}\mathcal{P}^\circ - 1\|^2 \le b^{-12} D_X$ and $\|\mathcal{K}_1\|^2 \le 2D_X$ (Problem 38.3.9). In the toy of Section 36.3 the second estimate fails (Problem 37.1.7); Problem 37.1.6 shows how a level repairs it.

The second moment is to be compared with the target. By Theorem 41.3.1,

$$d_T \le \#\bar G \cdot \deg \cdot \text{mass}_r.$$

Its proof compares the good forms on Y_0 with those on the definite space W_m; there the principal level is free, and the forms of the chosen weight have dimension exactly $\#\bar G \cdot \text{mass}_r \cdot \deg$. Problem 36.3.5 shows the correction at a level with stabilizers. With Theorem 40.3.1 this gives (Problem 41.3.6)

$$d_T \sum_{s\in\bar G} \|\mathcal{K}_s\|^2 \le \#\bar G\, (1+9b^{-4})\, D_X^2,$$

$1 + 9b^{-4}$ times the lower bound of Problem 9.2.5 for an exactly isometric projected map, with $\mathcal{Y}_0 = T$.

Good forms. Recall from Chapter 9 that $L\colon \mathcal{X} \to \mathcal{Y}$ is a near isometry with error ε if $\dim Z - \varepsilon \dim \mathcal{X} \le \|L \operatorname{pr}_Z\|^2$ for every subspace $Z \subset \mathcal{X}$, where pr_Z is the orthogonal projection onto Z, and that by Problem 9.3.1 this holds if $\varepsilon \ge 0$ and $\|L^*L - 1\|^2 \le \varepsilon^2 \dim \mathcal{X}$. By Theorem 41.2.2, $e_0\mathcal{P}^\circ$ is a near isometry with error $5b^{-4}$:

$$\|e_0\mathcal{P}^\circ \operatorname{pr}_Z\|^2 \ge \dim Z - 5b^{-4} D_X \qquad \text{for every subspace } Z \text{ of the coordinate space.}$$

Its proof writes the action of e_0 as a product of two commuting projections, one for each level prime, and bounds what each removes from $\mathcal{P}^\circ$ by a trace against ρ_{Bes}.

Few translates. Recall Theorems 9.3.1 and 9.4.1. *One translate*: let ρ be unitary, p the orthogonal projection onto a $\bar G$-stable subspace $\mathcal{Y}_0$, $\varepsilon \ge 0$, $p\mathcal{P}^\circ$ a near isometry with error ε, and $\dim \mathcal{Y}_0 \sum_s \|\mathcal{K}_s\|^2 \le \#\bar G\, (1+\varepsilon)(\dim \mathcal{X})^2$. Then for every subspace $\mathcal{F} \subset \mathcal{Y}_0$, with $\omega := \dim \mathcal{F}/\dim \mathcal{Y}_0$ if $\mathcal{Y}_0 \ne 0$ and $\omega := 0$ otherwise, some $s \in \bar G$ has $\dim\{x \in \mathcal{X} : p\rho(s)\mathcal{P}^\circ x \in \mathcal{F}\} \le \big(\frac{1+\omega}{2} + \frac{3}{2}\varepsilon\big) \dim \mathcal{X}$. *Few translates*: let $(B_s\colon \mathcal{X} \to \mathcal{Y}_0)_{s\in\mathcal{I}}$ be a family of linear maps between finite-dimensional complex spaces, indexed by a set $\mathcal{I}$, with $\mathcal{X} \ne 0 \ne \mathcal{Y}_0$ and $0 < \varepsilon < 1/16$, such that for every subspace $\mathcal{F} \subset \mathcal{Y}_0$ some s has $\dim B_s^{-1}(\mathcal{F}) \le \big(\frac{1+\omega}{2} + \frac{3}{2}\varepsilon\big) \dim \mathcal{X}$. Then there is a finite set $J \subset \mathcal{I}$ with

$$\#J \cdot \dim \mathcal{X} \le \tfrac{5}{2}\big(1 + \log \tfrac{1}{16\varepsilon}\big) \dim \mathcal{Y}_0, \qquad \dim \sum_{s\in J} B_s(\mathcal{X}) \ge (1-16\varepsilon) \dim \mathcal{Y}_0.$$

Problem 36.4.1. Let $\varepsilon := (8b^2)^{-1}$. (a) If $b \ge 9$, then $5b^{-4} \le \varepsilon$ and $9b^{-4} \le \varepsilon$. (b) If $b^2 \ge 2$, then $16\varepsilon = 2b^{-2}$ and $\frac{5}{2}\big(1 + \log(b^2/2)\big) \le 5\log(1+2b^2)$; if $b^2 > 2$, then $0 < \varepsilon < 1/16$.

By Problem 36.4.1(a), Theorems 41.2.2, 40.3.1 and 41.3.1 give the hypotheses of the first lemma with $\varepsilon = (8b^2)^{-1}$ and $\mathcal{Y}_0 = T$. Nonzero scalars and the isomorphism C do not change the dimensions of preimages.

Theorem 36.4.1 (one translate)**.** *For every subspace $\mathcal{F} \subset T$, with $\omega := \dim \mathcal{F}/d_T$ (and $\omega := 0$ if $d_T = 0$), there is $s \in \bar{G}$ such that the preimage of $\mathcal{F}$ under $e_0\rho(s)\mathcal{P}\colon S \to T$ has dimension at most $\big(\frac{1+\omega}{2} + \frac{3}{2}(8b^2)^{-1}\big)D_X$.*

Ramified forms. Theorem A of Chapter 42 concerns the ramified source forms S_{ram}, defined with the operators T_w, $w \mid \pi_N$, in Chapter 34; there are f such places, and $T_w^{N-1} = 1$. The proof shows $D_X > 0$, $\dim S_{\mathrm{ram}} \geq (1 - f/(N-1))D_X$ and $f/(N-1) \leq b^{-8}$.

Problem 36.4.2. For each $w \mid \pi_N$, $(N-1)^{-1}\sum_{i=0}^{N-2} T_w^i$ is the projection of S onto the fixed vectors of T_w along its other eigenspaces, and $\ker \sum_{i=0}^{N-2} T_w^i$ is the sum of the eigenspaces of T_w for eigenvalues $\neq 1$.

Problem 36.4.3. For linear maps $B_s\colon \mathcal{X} \to \mathcal{Y}$, a finite set J of indices and a subspace $Z \subset \mathcal{X}$, $\dim \sum_{s\in J} B_s(\mathcal{X}) \leq \dim \sum_{s\in J} B_s(Z) + \#J\,(\dim \mathcal{X} - \dim Z)$.

Problem 36.4.4. For $b \geq 2$, $5b^{-8}\log(1 + 2b^2) \leq 2b^{-2}$. Hence if $\#J \cdot D_X \leq 5d_T \log(1+2b^2)$ and $\dim S_{\mathrm{ram}} \geq (1-b^{-8})D_X$, then $\#J\,(D_X - \dim S_{\mathrm{ram}}) \leq 2b^{-2}d_T$.

The target is not zero, since the source is not zero and $e_0\mathcal{P}^\circ$ is a near isometry with error less than 1. Theorem 36.4.1, the second lemma with Problem 36.4.1(b), and Problems 36.4.3 and 36.4.4 prove the following.

Theorem 36.4.2 (Theorem B)**.** *For all data of Chapter 33, $d_T > 0$, and there is a finite set $J \subseteq \bar{G}$ with*

$$\#J \cdot \dim S_{\mathrm{ram}} \leq 5\,d_T \log(1+2b^2) \qquad \text{and} \qquad (1-4b^{-2})\,d_T \leq \dim \sum_{s\in J} e_0\rho(s)\mathcal{P}(S_{\mathrm{ram}}).$$

These maps $e_0\rho(s)\mathcal{P}$ on S_{ram} have the same ranges as the maps A_s of Chapter 42, which differ from them by the nonzero scalar $c_B c_{\mathrm{ss}}^{-1}\#\bar{G}$. Before restriction to S_{ram} the missing dimension is at most $2b^{-2}d_T$; the restriction costs at most a further $2b^{-2}d_T$. This is the step of the proof that supplies enough forms: Theorem 36.4.2 is the statement used, with Theorem A, in the final chapter.

In the Lean formalization. In the Lean proof [GL]: the Poincaré maps of Definition 36.2.1 are `Data.poincareAt`, with `inner_poincareAt`, and that of the fixed pair is `Data.poincare`; the scaled map is `Data.scaledPoincare`; ρ_{Bes} is `Filling.besselTranslate`, with `besselTranslate_eq_zero`, `besselTranslate_one`, `besselTranslate_mul` and `adjoint_besselTranslate` for its two properties; fillings are `Filling`, with `Filling.IsAdmissible` and `WeilData`. Theorem 36.4.1 follows from `isSpread_goodPoincare`, since `Data.goodPoincare` at s is $\#\bar{G}\,e_0\rho(s)\mathcal{P}$ and multiplication by $\#\bar{G}$ preserves the preimage of every subspace. The bounds for S_{ram} are `le_finrank_ramifiedSourceForms` and `f_div_N_sub_one_le`, and $D_X > 0$ is part of `Filling.Fits.finrank_levelSpace_eq_sourceDim`. Theorem 36.4.2 follows from `Data.enough_forms` for the normalized maps A_s, using `range_normalizedPoincare` and the nonzero factor $\#\bar{G}$.

Hints

§36.1.

36.1.1: Write $nv_{a+1} = v_{a+1} + \sum_{c \le a} x_c(n) v_c$, and compute the coefficients of $nn' v_{a+1}$; an element of $U(X)$ fixes every v_c and z.

36.1.2: An element of $U(X)$ preserves the span of the v_a and its orthogonal complement.

36.1.3: First the unipotent radical of the stabilizer of $Ev_1 + \cdots + Ev_\varrho$: $2\varrho(r - 2\varrho)$ dimensions from maps between the quotient, the middle and the isotropic subspace, and ϱ^2 from a skew-hermitian matrix over E. Then the unipotent radical of a Borel subgroup of $\mathrm{GL}_\varrho(E)$.

§36.2.

36.2.1: $\jmath\xi$ is fixed by Γ_X, and $\langle \psi, \gamma v \rangle = \langle \gamma^{-1}\psi, v \rangle$.

36.2.2: Problem 36.2.1.

36.2.3: Apply Problem 36.2.1 with $\psi = P\xi$, and group the cosets $\gamma\Gamma_X$ by double cosets.

§36.3.

36.3.1: Multiply $\mathrm{diag}(u_1, u_2)$ and the antidiagonal matrices by $\mathrm{diag}(u, 1)$ on both sides.

36.3.2: Representatives of Γ/Γ_X: $\mathrm{diag}(1, u)$ and $\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}$; sum the powers of u.

36.3.3: Problem 21.3.2 for c_0; then Problems 36.2.3 and 36.3.1.

36.3.4: Problem 21.3.5.

36.3.5: Write $a = 8q$ or $a = 8q + 4$.

§36.4.

36.4.1: For (b), $\log(1 + 2b^2) \ge 1$ and $b^2/2 \le 1 + 2b^2$.

36.4.2: T_w is diagonalizable.

36.4.3: Choose a complement of Z in $\mathcal{X}$.

36.4.4: $\log(1 + 2b^2) \le 2b^2$.

References

[GGP] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups*, Astérisque **346** (2012), 1–109.

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.