DIVISIBILITY ALONG THE CHAIN

Prerequisites. Formal power series over a commutative ring. Finite extensions of $\mathbb{Q}_p$, their valuations, and lattices over their rings of integers; the length of a module of finite length. Gaussian binomial coefficients $\begin{bmatrix} n \\ l \end{bmatrix}_Q$ [Mac, Ch. I, §2, Example 3]. From Chapter 5, *Size against divisibility*: that a sublattice of small colength has small index, so that a gain on it is charged its colength. From Chapter 8, *The chain and the square root*: Section 8.4. From Chapter 18, *Hecke nondegeneracy on the tree*: the loss, defined in Section 18.1. From Chapter 24, *Hecke operators and theta lifts*: Section 24.5, in particular Problems 24.5.3 and 24.5.4. Section 25.6 says where the proof uses this transfer; the step in the generality of the proof is Theorem 53.4.1 of Chapter 53, and the rest of the chapter does not use it.

Problem 8.4.3 adds the divisibility exponents that hold on the whole lattice at each step of the chain. In the proof, Theorem 53.4.1 bounds the gauge of the restricted chain on the image of the integral source forms under a power of a product of Hecke operators. The local model in Chapter 18 measures the colength of the values of a Hecke operator at a single lattice. How does a gain on a sublattice at the start of a chain become a gain of the composite? In the model of Problem 24.5.4, moving the source operator through the chain introduces signed complete symmetric functions. Their valuations enter the bound in Section 25.4. The model below uses one Hecke operator. Theorem 53.4.1 uses a product indexed by a full mark above p.

25.1. Elementary and complete symmetric functions

Let $x = (x_1, \dots, x_n)$ be a finite family in a commutative ring. The *elementary* and *complete symmetric functions* $e_l(x)$ and $h_l(x)$ are the sums of the products $x_{i_1} \cdots x_{i_l}$ over $i_1 < \cdots < i_l$ and over $i_1 \le \cdots \le i_l$ respectively, with $e_0 = h_0 = 1$. Their generating series are

$$E_x(X) := \prod_i (1+x_iX) = \sum_{l \ge 0} e_l(x) X^l, \qquad H_x(X) := \prod_i (1-x_iX)^{-1} = \sum_{l \ge 0} h_l(x) X^l.$$

All degrees l are nonnegative integers, and $\begin{bmatrix} n \\ l \end{bmatrix}_Q = 0$ for $l > n$.

Problem 25.1.1. $E_x(X)H_x(-X) = 1$, that is, $\sum_{l=0}^{r} (-1)^l e_{r-l}(x)\, h_l(x) = 0$ for $r \ge 1$. For $x = (a, b)$, $h_2(x) = a^2 + ab + b^2$.

This is [Mac, Ch. I, (2.6)]. For families x and y write $x \sqcup y$ for the concatenated family.

Problem 25.1.2.

$$e_r(x \sqcup y) = \sum_{l=0}^{r} e_l(x)\, e_{r-l}(y), \qquad h_r(x \sqcup y) = \sum_{l=0}^{r} h_l(x)\, h_{r-l}(y).$$

The families that occur are geometric progressions.

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Problem 25.1.3. For $t \geq 0$,

$$e_l(1, Q, \ldots, Q^t) = Q^{l(l-1)/2} \begin{bmatrix} t+1 \\ l \end{bmatrix}_Q, \qquad h_l(1, Q, \ldots, Q^t) = \begin{bmatrix} t+l \\ l \end{bmatrix}_Q.$$

For instance $h_2(1, Q) = 1 + Q + Q^2$.

These are [Mac, Ch. I, §2, Example 3]. Let q be invertible, $Q := q^2$, $g \geq 1$ and $t := g - 1$. A *block* is a family

$$x = \beta\,(q^t, q^{t-2}, \ldots, q^{-t}) \tag{1}$$

of g terms with ratio Q^{-1}, where β is invertible. For $\beta = 1$ this is the family in Problem 24.5.3, where q^t is written $Q^{t/2}$.

Problem 25.1.4. For the block (1), $e_l(x) = \beta^l q^{-l(g-l)} \begin{bmatrix} g \\ l \end{bmatrix}_Q$ and $h_l(x) = \beta^l q^{-lt} \begin{bmatrix} t+l \\ l \end{bmatrix}_Q$.

For $\beta = 1$ the first formula, in degrees $0 \leq l \leq g$, is Problem 24.5.3.

25.2. Valuations

Let k be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and valuation v normalized by $v(p) = 1$.

Problem 25.2.1. $\begin{bmatrix} n \\ l \end{bmatrix}_Q$ is a polynomial in Q with integer coefficients and constant term 1, for $0 \leq l \leq n$. Hence $v\big(\begin{bmatrix} n \\ l \end{bmatrix}_Q\big) = 0$ if $Q \in k$ and $v(Q) > 0$.

Problem 25.2.2. Let $a \in k^\times$ and $Q \in k^\times$ with $v(Q) = v_Q > 0$. Then

$$\begin{aligned} v\big(e_l(a, aQ, \ldots, aQ^t)\big) &= l\,v(a) + v_Q\,\frac{l(l-1)}{2} && (0 \leq l \leq t+1), \\ v\big(h_l(a, aQ, \ldots, aQ^t)\big) &= l\,v(a) && (l \geq 0). \end{aligned}$$

For $0 \leq l \leq t+1$, the difference $v(e_l) - v(h_l)$ for this family is $v_Q\,l(l-1)/2$.

Problem 25.2.3. Let x be the block (1) with $\beta, q \in k^\times$ and $v(q) = \nu > 0$. Then $v(h_l(x)) = l\,(v(\beta) - t\nu)$ for $l \geq 0$, and $v(e_l(x)) = l\,(v(\beta) - t\nu) + \nu\,l(l-1)$ for $0 \leq l \leq g$. In particular every $h_l(x)$ lies in $\mathcal{O}$ if and only if $v(\beta) \geq t\nu$.

For a union of blocks only a bound survives.

Problem 25.2.4. Let Z be a finite family in k whose least valuation is a real number μ. Then $v(h_l(Z)) \geq l\mu$, with equality if exactly one member of Z has valuation μ. For $Z = (1, 1)$ and $Z = (1, -1)$ one has $h_l(Z) = l + 1$ and $h_l(Z) = (1 + (-1)^l)/2$.

Problem 25.2.5. Let $Z = x_1 \sqcup x_2$, where $x_i = \beta_i\,(q, q^{-1})$ are blocks with $g = 2$, and let $v(q) = \nu > 0$ and $v(\beta_1) = v(\beta_2) = 0$. Then $h_1(Z) = (\beta_1 + \beta_2)(q + q^{-1})$, so $v(h_1(Z)) = v(\beta_1 + \beta_2) - \nu$, which exceeds $-\nu$ exactly when $\beta_1 \equiv -\beta_2$ modulo the maximal ideal of $\mathcal{O}$.

25.3. Moving a Hecke operator along a chain

We use the setting of Problem 24.5.4. Let $F_0, \dots, F_m$ be finite-dimensional k-vector spaces, let $K_j \colon F_j \to F_{j+1}$ be linear maps, and let $\mathcal{K}_j := K_{j-1} \cdots K_0$, with $\mathcal{K}_0 := 1$. Fix $r \geq 1$. For each j let $\mathcal{E}_0^{(j)} = 1, \mathcal{E}_1^{(j)}, \dots, \mathcal{E}_r^{(j)}$ be operators on F_j, and let $x_1, \dots, x_m$ be finite families in k with

$$\mathcal{E}_i^{(j+1)} K_j = K_j \sum_{l=0}^{i} e_l(x_{j+1})\, \mathcal{E}_{i-l}^{(j)} \qquad (0 \leq i \leq r,\ 0 \leq j < m). \tag{2}$$

This is the case with source argument X of the split relation in Section 24.5. Under the hypotheses of Theorem 53.4.1, statement (B) of Chapter 53 gives this relation for every translation in the support of the datum, at every prime of a mark, on forms spherical there, for $r \leq N_0$. The family at each step is a block (1). Let $Z_j := x_1 \sqcup \cdots \sqcup x_j$. In (2) the operators at the target are expressed through those at the source. We need the converse.

Problem 25.3.1. Let $r = 1$ and $m \geq 2$. Then $K_0 \mathcal{E}_1^{(0)} = \big(\mathcal{E}_1^{(1)} - e_1(x_1)\big) K_0$, and $\mathcal{K}_2 \mathcal{E}_1^{(0)} = \big(\mathcal{E}_1^{(2)} - h_1(Z_2)\big) \mathcal{K}_2$.

Problem 25.3.2. Let $r = 2$ and $m \geq 1$. Then $K_0 \mathcal{E}_2^{(0)} = \big(\mathcal{E}_2^{(1)} - h_1(x_1)\, \mathcal{E}_1^{(1)} + h_2(x_1)\big) K_0$.

Problem 25.3.3. $\mathcal{E}_i^{(j)}\, \mathcal{K}_j = \mathcal{K}_j \sum_{l=0}^{i} e_l(Z_j)\, \mathcal{E}_{i-l}^{(0)}$ for $0 \leq i \leq r$.

Problem 24.5.4, applied to the first j maps, gives the following identity.

Theorem 25.3.1. *For $1 \leq j \leq m$, $\mathcal{K}_j\, \mathcal{E}_r^{(0)} = \displaystyle\sum_{l=0}^{r} (-1)^l h_l(Z_j)\, \mathcal{E}_{r-l}^{(j)}\, \mathcal{K}_j$.*

The relation for one lift has the elementary functions of a block. For a block x with $v(q) = \nu > 0$ and $0 \leq l \leq g$, one has $v(e_l(x)) - v(h_l(x)) = \nu\, l(l-1)$ (Problem 25.2.3). The inverse relation has the complete functions, which may have denominators. For the lattice normalization in the next two problems assume $1 \leq r \leq N_0$, where N_0 is defined below. The operators $\mathcal{E}_i^{(j)}$ are normalized forms of the sums $T_i^{(j)}$ over edges of type i of Section 24.5: when F_j consists of functions of lattices in a space of dimension N_j, with $N_0 < N_1 < \cdots < N_m$, one takes $\mathcal{E}_i^{(j)} = q^{-i(N_j - i)}\, T_i^{(j)}$ for $0 \leq i \leq N_j$.

Problem 25.3.4. With this normalization, Theorem 25.3.1 reads

$$\sum_{l=0}^{r} (-1)^l h_l(Z_j)\, q^{l(N_j - 2r + l)}\, T_{r-l}^{(j)}\, \mathcal{K}_j = q^{r(N_j - N_0)}\, \mathcal{K}_j\, T_r^{(0)}.$$

Problem 25.3.5. Suppose that Z_j is a union of blocks (1) with the same t, that $v(q) > 0$, that $v(\beta) \geq 0$ for each block, and that $N_j \geq 2r + t - 1$. Then every coefficient $h_l(Z_j)\, q^{l(N_j - 2r + l)}$ in Problem 25.3.4 lies in $\mathcal{O}$.

25.4. A gain on a sublattice

For each j let $\mathrm{Int}_j \subset F_j$ be an $\mathcal{O}$-lattice of full rank. For $w \in F_j$ let $\mathrm{ord}(w)$ be the least valuation of a coordinate of w in a basis of Int_j, with $\mathrm{ord}(0) = \infty$; we use $v(0) = +\infty$ and $\min \varnothing = +\infty$. For a linear map $\phi \colon F_i \to F_j$ let $\mathrm{gain}(\phi)$ be the

least value of $\mathrm{ord}(\phi y)$ over the members y of a basis of Int_i. For $k = \mathbb{Q}_p$ and $\phi \neq 0$, $\mathrm{gain}(\phi)$ is the largest integer n with $\phi(\mathrm{Int}_i) \subseteq p^n \mathrm{Int}_j$; so $\mathrm{gain}(\phi) \log p$ plays at p the role of the divisibility $-h_{\mathrm{fin}}$ of Section 8.4.

Problem 25.4.1. ord and gain do not depend on the bases. $\mathrm{ord}(\phi w) \geq \mathrm{ord}(w) + \mathrm{gain}(\phi)$, $\mathrm{gain}(\phi\psi) \geq \mathrm{gain}(\phi) + \mathrm{gain}(\psi)$, $\mathrm{gain}(\phi + \psi) \geq \min(\mathrm{gain}(\phi), \mathrm{gain}(\psi))$, and $\mathrm{gain}(c\phi) = v(c) + \mathrm{gain}(\phi)$ for $c \in k^\times$.

Assume (2). For $1 \leq j \leq m$ and $0 \leq i \leq r$ let

$$\gamma_j(i) := \mathrm{gain}\big(\mathcal{E}_i^{(j)} K_{j-1}\big),$$

so that $\gamma_j(0) = \mathrm{gain}(K_{j-1})$ is the gain of the jth step on the whole lattice, and $\gamma_j(r)$ its gain after the Hecke operator $\mathcal{E}_r^{(j)}$ at the target. Let $c \in k^\times$ be such that $Y := c\,\mathcal{E}_r^{(0)}$ maps Int_0 into itself, and let $A := v(c)$. To bound $\mathrm{ord}(\mathcal{K}_j\, \mathcal{E}_r^{(0)} w)$ in terms of $\mathrm{ord}(\mathcal{K}_{j-1} w)$, define

$$\delta_j := \min_{0 \leq l \leq r} \big(v(h_l(Z_j)) + \gamma_j(r-l)\big).$$

When $\gamma_j(r) < +\infty$, the *price* of the jth step is $\mathrm{Pr}_j := \gamma_j(r) - \delta_j$. It is nonnegative.

Problem 25.4.2. For $w \in F_0$ and $1 \leq j \leq m$, $\mathrm{ord}\big(\mathcal{K}_j\, \mathcal{E}_r^{(0)} w\big) \geq \mathrm{ord}\big(\mathcal{K}_{j-1} w\big) + \delta_j$.

Problem 25.4.3. Let $S \subseteq \{1, \ldots, m\}$, and let n_j be the number of elements of S that are at most j. For $G \in \mathrm{Int}_0$ and $0 \leq j \leq m$,

$$\mathrm{ord}\big(\mathcal{K}_j\, Y^{n_j} G\big) \geq \sum_{1 \leq i \leq j,\ i \in S} \big(A + \delta_i\big) + \sum_{1 \leq i \leq j,\ i \notin S} \gamma_i(0).$$

Theorem 25.4.1. *For every $F \in \mathrm{Int}_0$,*

$$\mathrm{ord}\big(\mathcal{K}_m\, Y^m F\big) \geq \sum_{j=1}^{m} \max\big(\gamma_j(0),\ A + \delta_j\big).$$

The gain holds on the image $Y^m \mathrm{Int}_0$. When Y is injective this is a sublattice of full rank, and in Chapter 5 the gain on such a sublattice is charged its colength. In the proof one copy of the product $Y_{\mathfrak{M}}$ of Hecke operators of a full mark (Section 25.6) goes into the datum of every lift, and the transfer is cleared by a power of p fixed in advance (Problem 53.4.2).

Problem 25.4.4. If Y is injective, then the length of $\mathrm{Int}_0 / Y^m \mathrm{Int}_0$ is m times the length of $\mathrm{Int}_0 / Y\, \mathrm{Int}_0$.

In the tree model of Chapter 18, as F ranges over all integral forms the values $(YF)(L)$ fill $S(L)$, and the loss at L is the length of $H(L)/S(L)$. The proof does not assume that Y is injective; the colength bound it uses is in Chapter 48, Section 48.4.

25.5. Stepwise maxima overcount

Theorem 25.4.1 uses δ_j at step j. Can one replace its lower bound for $\mathrm{gain}(\mathcal{K}_m Y^m)$ by $\sum_{j=1}^m \max(\gamma_j(0), A + \gamma_j(r))$? We compare two chains with the same numbers $\gamma_j(i)$. Let p be any prime, $k = \mathbb{Q}_p$, $m = 2$, $r = 1$, $F_j = \mathbb{Q}_p^2$ and $\mathrm{Int}_j = \mathbb{Z}_p^2$. Let $K_0 = K_1 = K$ be a diagonal matrix, let $x_1 = (\alpha_1)$ and $x_2 = (\alpha_2)$ be families of one element, and let

$$Y := \mathcal{E}_1^{(0)} := \mathrm{diag}(p, 1), \qquad \mathcal{E}_1^{(j)} := Y + h_1(Z_j) \quad (j = 1, 2),$$

so that $A = 0$.

Problem 25.5.1. These data satisfy (2), and length $\mathbb{Z}_p^2/Y\mathbb{Z}_p^2 = 1$.

Problem 25.5.2. Let $K = \mathrm{diag}(1, p)$ and $\alpha_1 = \alpha_2 = p$. Then $\gamma_j(0) = 0$, $\gamma_j(1) = 1$ and $\mathrm{Pr}_j = 0$ for $j = 1, 2$, and $\mathrm{gain}(\mathcal{K}_2\, Y^2) = 2$.

Problem 25.5.3. Let $K = \mathrm{diag}(p, 1)$, $\alpha_1 = p - 1$ and $\alpha_2 = -2p$. Then $\gamma_j(0) = 0$, $\gamma_j(1) = 1$ and $\mathrm{Pr}_j = 1$ for $j = 1, 2$, and $\mathrm{gain}(\mathcal{K}_2\, Y^2) = 0$.

The sum of the stepwise maxima is 2 in both chains. Theorem 25.4.1 gives the exact gain in each case.

Theorem 25.4.1 uses the gains $\gamma_j(i)$, the scaling valuation A, and the coefficients $h_l(Z_j)$. Chapters 15 and 16 model divisibility for a single theta kernel. If Z_j has finite least valuation μ_j, then $v(h_l(Z_j)) \geq l\mu_j$ (Problem 25.2.4). A separate colength estimate enters when the gain on the image is used; Chapter 18 computes a local colength at one lattice.

Exercises.

(1) The Newton polygon of $\prod_{i=0}^{t}(1 + aQ^iX)$ over k, with $a \in k^\times$, $Q \in k^\times$ and $v(Q) > 0$, has the vertices $\big(l, l\,v(a) + v(Q)\,l(l-1)/2\big)$ for $0 \leq l \leq t+1$.

(2) Suppose that (2) holds with $u_{j+1}^i K_j$ in place of K_j on the right, for units $u_j \in \mathcal{O}^\times$. Write $E_j(X) := \sum_{i=0}^{r} \mathcal{E}_i^{(j)} X^i$. Show first that the relation is equivalent modulo X^{r+1} to $E_{j+1}(X)K_j = K_j E_{x_{j+1}}(u_{j+1}X)E_j(u_{j+1}X)$, the relation of Section 24.5 with $P(X) = E_{u_{j+1}x_{j+1}}(X)$. Then Theorem 25.3.1 holds with $\mathcal{K}_j\mathcal{E}_r^{(0)}$ multiplied by $(u_1 \cdots u_j)^r$, and with Z_j replaced by the family Z_j' of the $u_l u_{l+1} \cdots u_j\, a$ for $l \leq j$ and a in x_l. Theorem 25.4.1 holds with δ_j computed from Z_j'. For the same gains $\gamma_j(i)$, the two values of δ_j agree when Z_j has exactly one member of least valuation. For $r = 1$, p odd, $x_1 = (1)$, $x_2 = (-1)$, $u_1 = -1$, $u_2 = 1$ and $\gamma_2(1) > \gamma_2(0)$, the two values of δ_2 differ.

25.6. The step in the proof

At an exceptional prime $p < x_1$ the proof [GL] applies this transfer to the Hecke operators of a full mark and to the chain of m lifts, and obtains Theorem 53.4.1; Chapter 53 states and develops it, and Chapter 48, Section 48.4, uses it. There the relation (2) appears for the lift $X_j \to X_{j+1}$ with one polynomial for all translations, in degrees $i \leq r_p$, with a block $\chi(q^t, q^{t-2}, \ldots, q^{-t})$ in place of x_{j+1}, χ a p-adic unit (this comparison is ours); the proof inverts that polynomial over the ring $\mathcal{O}_\eta$ of Chapter 34 and puts one copy of the product of Hecke operators into the datum of every lift, as in the choice $S = \{1, \ldots, m\}$ of Problem 25.4.3.

Hints

§25.1.

25.1.1: Compare the product formulas.

25.1.2: $E_{x \sqcup y} = E_x E_y$.

25.1.3: Induction on t and l together, using $\begin{bmatrix} n \\ 0 \end{bmatrix}_Q = 1$ at the boundary and, for $1 \leq l \leq n$, the rule $\begin{bmatrix} n \\ l \end{bmatrix}_Q = \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_Q + Q^l \begin{bmatrix} n-1 \\ l \end{bmatrix}_Q$ and its companion $\begin{bmatrix} n \\ l \end{bmatrix}_Q = Q^{n-l} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_Q + \begin{bmatrix} n-1 \\ l \end{bmatrix}_Q$.

25.1.4: Problem 25.1.3 for the family $\beta q^{-t}(1, Q, \ldots, Q^t)$; e_l and h_l are homogeneous of degree l.

§25.2.

25.2.1: The first rule in the hint to Problem 25.1.3.
25.2.2: Problems 25.1.3 and 25.2.1.
25.2.3: Problems 25.1.4 and 25.2.1.
25.2.4: Which monomial of degree l has the least valuation?
25.2.5: Problem 25.1.2.

§25.3.

25.3.1: Apply (2) with $i = 1$ once for each step.
25.3.2: Use (2) with $i = 1$ to remove $K_0\mathcal{E}_1^{(0)}$, then Problem 25.1.1.
25.3.3: Write (2) as an identity of polynomials in X modulo X^{r+1}, as in the hint to Problem 24.5.4, and use Problem 25.1.2.
25.3.4: Multiply Theorem 25.3.1 by $q^{r(N_j-r)}$.
25.3.5: Problems 25.2.3 and 25.2.4; look at $l = 1$.

§25.4.

25.4.1: Write w in coordinates.
25.4.2: Apply Theorem 25.3.1 to w, write $\mathcal{K}_j = K_{j-1}\mathcal{K}_{j-1}$, and use Problem 25.4.1.
25.4.3: Induction on j. For $j \in S$ write $Y^{n_j} = c\,\mathcal{E}_r^{(0)}\,Y^{n_{j-1}}$ and use Problem 25.4.2. For Theorem 25.4.1, compare $A + \delta_j$ with $\gamma_j(0)$ when choosing S, and apply the problem to $Y^{m-n_m}F$.
25.4.4: $Y^i\mathrm{Int}_0/Y^{i+1}\mathrm{Int}_0 \cong \mathrm{Int}_0/Y\,\mathrm{Int}_0$.

§25.5.

25.5.1: Diagonal matrices commute.
25.5.2: Compute $\mathcal{K}_2\,Y^2$.
25.5.3: Compute $\mathcal{E}_1^{(1)}$ and $\mathcal{E}_1^{(2)}$ first.

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[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.