HECKE OPERATORS AND THETA LIFTS

Prerequisites. Even unimodular lattices, E_8, theta series and the Hecke operators T_n on modular forms of level one [SeA, Ch. V and VII]. Quadratic forms over $\mathbb{F}_p$ and the number of their isotropic vectors (for split forms, Problem 22.2.2). Gaussian binomial coefficients $\begin{bmatrix} a \\ b \end{bmatrix}_Q$, which count the b-dimensional subspaces of $\mathbb{F}_Q^a$ [Andr, Ch. 3]. From Chapter 21, *Masses and the count of forms*: Section 21.1, in particular Problems 21.1.1, 21.1.2 and 21.2.3. From Chapter 22, *The Siegel–Weil formula in the smallest case*: Problems 22.1.2 and 22.3.7. From Chapter 23, *Definite theta lifts and their norms*: Section 23.1. For Section 24.4, maximal orders in a definite quaternion algebra over $\mathbb{Q}$ [Gr], [Pi].

A theta lift is a sum over classes of lattices. A Hecke operator on a class set is also a sum over lattices, namely over the neighbours of a lattice. How do the two interact? Eichler found that a theta series summed over the neighbours of a lattice is a Hecke transform of the theta series of the lattice, up to an explicit multiple of the identity [Ei52], [Ei73]; see [An] for Siegel modular forms and [Ra] for the general theta correspondence. We prove this in degree one, and check it on class sets with two elements, where the Hecke operator is a 2×2 matrix.

24.1. Neighbours

How does one move from a lattice to the other lattices of its genus? Kneser's answer is to change it at one prime [Kn].

Let L be a positive definite even unimodular lattice of positive rank $n = 2k$, with $8 \mid n$, and let p be a prime. A *p-neighbour* of L is an even unimodular lattice $M \subset L \otimes \mathbb{Q}$ with $[L : L \cap M] = [M : L \cap M] = p$. Let $\mathcal{N}_p(L)$ be the set of p-neighbours.

Problem 24.1.1. Let p be odd. For $v \in L \setminus pL$ with $v \cdot v \equiv 0 \pmod{2p^2}$, let $L_v := \{x \in L : x \cdot v \equiv 0 \pmod{p}\}$. Then $M_v := L_v + \mathbb{Z}\, v/p$ is a p-neighbour of L, and M_v depends only on the line $\mathbb{F}_p \bar{v} \subset L/pL$. Every p-neighbour arises in this way, from exactly one isotropic line of L/pL.

Problem 24.1.2. For odd p, the quadratic space L/pL over $\mathbb{F}_p$ is split, so that

$$|\mathcal{N}_p(L)| = \frac{(p^k - 1)(p^{k-1} + 1)}{p - 1}.$$

For $L = E_8$ and $p = 3, 5$ this is 1120 and 19656.

For $p = 2$ the same count holds, if v is chosen with $v \cdot v \equiv 0 \pmod 8$ [CL]; for E_8 it is 135. Since E_8 is the only even unimodular lattice of rank 8, all its neighbours are copies of E_8. In each even unimodular genus considered here, the graph on isometry classes defined by p-neighbours is connected for every prime p, by Kneser's theorem [Kn], [CL].

Let Cl be the set of isometry classes in the genus of L, and let $\Gamma_L := \mathrm{O}(L)$. The *neighbour operator* on functions $\varphi\colon \mathrm{Cl} \to \mathbb{C}$ is $(T_p\varphi)(L) := \sum_{M \in \mathcal{N}_p(L)} \varphi(M)$, and

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its matrix $N_p(L,M) := \#\{M' \in \mathcal{N}_p(L) : M' \cong M\}$ is the *Brandt matrix* of the class set at p. Let $\langle \varphi, \psi \rangle := \sum_{L \in \mathrm{Cl}} \varphi(L)\overline{\psi(L)}/|\Gamma_L|$, as in Chapter 21.

Problem 24.1.3. $N_p(L,M)/|\Gamma_L| = N_p(M,L)/|\Gamma_M|$. Hence T_p is self-adjoint for $\langle\,,\rangle$, and the constant function is an eigenvector with eigenvalue $|\mathcal{N}_p(L)|$.

In rank 16 the class set is $\{E_8 \perp E_8,\ D_{16}^+\}$, with $|\Gamma| = 2 \cdot 696729600^2$ and $2^{15} \cdot 16!$ [CoS].

Problem 24.1.4. The Brandt matrix of the class set of rank 16 at $p = 2$ is $\begin{pmatrix} 14670 & 18225 \\ 12870 & 20025 \end{pmatrix}$, with eigenvalues 32895 and 1800.

The eigenvalue 1800 is $15\,\tau(2) + (2^4 + 2^5 + \cdots + 2^{10}) + 2^7$, where $\tau(2) = -24$ is the coefficient of Ramanujan's Δ. Chenevier and Lannes explain such identities by the Arthur parameters of the eigenvectors [CL]. In Section 24.2 we show that the theta series of degree one cannot see this eigenvector.

24.2. Eichler's commutation relation in degree one

For L as in Section 24.1, let $r_L(m) := \#\{x \in L : x \cdot x = 2m\}$, with $r_L(m) := 0$ unless $m \in \mathbb{Z}_{\geq 0}$, and let $\theta_L := \sum_m r_L(m)\, q^m$. It is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ [SeA, Ch. VII], and T_{p^2} acts on its coefficients by

$$a(m) \mapsto a(p^2 m) + p^{k-1} a(m)\,[p \mid m] + p^{2k-2} a(m/p^2).$$

For the remainder of the neighbour count, let p be odd.

Problem 24.2.1. Let $x \in L \setminus pL$ with $x \cdot x = 2m$. The number of $M \in \mathcal{N}_p(L)$ with $x \in M$ is $(p^{2k-2} - 1)/(p-1)$ if $p \nmid m$, and $1 + p(p^{k-1} - 1)(p^{k-2} + 1)/(p-1)$ if $p \mid m$.

Problem 24.2.2. The vectors of $\bigcup_{M \in \mathcal{N}_p(L)} M \setminus L$ are the y/p with $y \in L \setminus pL$ and $y \cdot y \equiv 0 \pmod{2p^2}$, and each lies in exactly one neighbour. Deduce Theorem 24.2.1.

Theorem 24.2.1 (Eichler)**.** *For odd p,* $$\sum_{M \in \mathcal{N}_p(L)} \theta_M = T_{p^2}\theta_L + \frac{p^{2k-2} - p}{p-1}\,\theta_L.$$

Problem 24.2.3. The constant-term identity in Theorem 24.2.1 for $L = E_8$ and $p = 3$ reads $1120 = 757 + 363$. The same identity of constant terms holds for $p = 2$: $135 = 73 + 62$.

Let $\theta(\varphi) := \sum_{L \in \mathrm{Cl}} \varphi(L)\,\theta_L/|\Gamma_L|$ be the *theta lift* of a function on the class set. It is the analogue for these lattices of the lift of Chapter 23 to rank one in weight zero, with the vectors sorted by norm into a q-series instead of weighted by a Gaussian.

Problem 24.2.4. Let p be odd. Then $\theta(T_p\varphi) = \big(T_{p^2} + \frac{p^{2k-2}-p}{p-1}\big)\theta(\varphi)$. If $\theta(\varphi) \neq 0$ and $T_p\varphi = \mu\varphi$, and $\theta(\varphi)$ is an eigenform with $T_p\theta(\varphi) = \lambda\theta(\varphi)$, then $\mu = \lambda^2 - p^{k-1} + (p^{2k-2} - p)/(p-1)$.

For odd p, the theta lift intertwines the neighbour operator with $T_p^2 - p^{k-1} + (p^{2k-2} - p)/(p-1)$ on modular forms.

In rank 16, $\theta_{E_8 \perp E_8} = \theta_{D_{16}^+}$ by Problem 22.1.2 of Chapter 22.

Problem 24.2.5. The eigenvector of Problem 24.1.4 with eigenvalue 1800 lies in the kernel of θ. For the constant function, the eigenvalue 32895 agrees with the formula of Problem 24.2.4 read at $p = 2$, where we have not proved it: $129^2 - 2^7 + (2^{14} - 2) = 32895$.

The two lattices are distinguished by their theta series of degree 4, and not before [CL]. To detect this eigenspace by theta series one can pass to higher degree; for lift norms see Section 23.3.

24.3. Hermitian lattices of rank five

Let $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$, let $p \equiv 1 \pmod 4$ be a prime, and let $p = \pi\bar\pi$ in $\mathcal{O}$. Let L be a unimodular Hermitian lattice of rank $n \geq 1$, as in Chapter 21, and let $L^\sharp$ denote duals. Then $L/\pi L$ is an n-dimensional vector space over $\mathcal{O}/\pi = \mathbb{F}_p$. For a hyperplane $H \subset L/\pi L$ let L_H be its preimage in L, and

$$L'_H := L_H + \pi L_H^\sharp.$$

Problem 24.3.1. L'_H is unimodular and lies in the genus of L, with $L \cap L'_H = L_H$ of index p in both. Moreover $L'_H \subset \bar\pi^{-1}L$, and L is obtained from L'_H by the same construction with $\bar\pi$ in place of π.

We call L'_H a *π-neighbour* of L. There are $(p^n - 1)/(p - 1)$ of them. For L representing a class in Cl_n, the genus of I_n in Chapter 21, set $T_\pi\varphi(L) := \sum_H \varphi(L'_H)$. For the mass inner product, the adjoint of T_π is $T_{\bar\pi}$. Let $r_L(m) := \#\{x \in L : h(x,x) = m\}$ for $m \in \mathbb{Z}_{\geq 0}$ and $r_L(m) := 0$ otherwise, and let $\theta_L := \sum_m r_L(m)\, q^m$, a modular form of weight n for $\Gamma_0(4)$ with character χ^n, where χ is the character modulo 4 [Iw, Ch. 10]. Since $\chi(p) = 1$, T_p acts on such forms by $a(m) \mapsto a(pm) + p^{n-1}a(m/p)$.

Problem 24.3.2. For $x \in L$, the number of H with $x \in L'_H$ is $(p^{n-1} - 1)/(p - 1)$ if $x \notin \pi L$, and $(p^n - 1)/(p - 1)$ if $x \in \pi L$. A vector $x \notin L$ lies in some L'_H exactly when $y := \bar\pi x \in L \setminus \bar\pi L$ and $p \mid h(y,y)$, and then in exactly one.

Problem 24.3.3. $\displaystyle\sum_H \theta_{L'_H} = T_p\theta_L + \frac{p^{n-1} - p}{p - 1}\,\theta_L$. Hence, with $\theta(\varphi) := \sum_{L \in \mathrm{Cl}_n} \varphi(L)\theta_L/|\Gamma_L|$, one has $\theta(T_\pi\varphi) = \big(T_p + \frac{p^{n-1}-p}{p-1}\big)\theta(\varphi)$, and likewise for $\bar\pi$.

Here the factor is linear in T_p. At p the unitary group is $\mathrm{GL}_n(\mathbb{Q}_p)$. In the component corresponding to π, L'_H and L form an edge of type one in the sense of Section 24.5.

Now take $n = 5$, $p = 5$ and $\pi = 2 + \mathrm{i}$. By Problem 21.2.3, $\mathrm{Cl}_5 = \{I_5, \Lambda \perp I_1\}$, where Λ is the Gaussian Hermitian lattice of Section 21.1 with underlying real lattice E_8. By Problems 21.1.2 and 21.2.3 and $|\Gamma_\Lambda| = 46080$ (loc. cit.), $|\Gamma| = 122880$ and 184320, and $r(1) = 20$ and 4.

Problem 24.3.4. The matrix of T_π on Cl_5 is $\begin{pmatrix} 525 & 256 \\ 384 & 397 \end{pmatrix}$, with eigenvalues 781 and 141.

Let g be the cusp form called f in Chapter 22,

$$g = \frac{1}{4}\sum_{\alpha \in \mathcal{O}} \alpha^4 q^{|\alpha|^2} = \eta(\tau)^4\eta(2\tau)^2\eta(4\tau)^4 = q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + 81q^9 + \cdots;$$

here η is Dedekind's function. The form g spans the cusp forms of weight 5 for $\Gamma_0(4)$ with character χ, and it has complex multiplication by $\mathbb{Q}(\mathrm{i})$. By Problem 22.3.7, $\theta_{I_5} - \theta_{\Lambda\perp I_1} = 16\,g$.

Problem 24.3.5. The 141-eigenspace of T_π is spanned by $\varphi = (2,-3)$, and $\theta(\varphi) = g/3840$.

Problem 24.3.6. $a_5(g) = -14 = 2\,\mathrm{Re}\big((2+\mathrm{i})^4\big)$, and $141 = -14 + 155$, as Problem 24.3.3 predicts.

The lift identifies the two class functions supported on single classes, with value 1 there, with the basis $(\theta_{I_5}/122880, \theta_{\Lambda\perp I_1}/184320)$ of its image. In this basis the matrix of $T_5 + 155$ is the matrix of T_π in Problem 24.3.4. Neither theta series is an eigenform; the eigenforms are g and the lift of the constant function, with T_5-eigenvalues -14 and $626 = 781 - 155$.

24.4. Brandt matrices

The original setting of Eichler's relation is a quaternion algebra, where the theta series of the ideal lattices between ideal-class representatives give modular forms of weight two and prime level [Ei73], [Pi], [Gr]. We follow Gross [Gr, §§1–2] in the smallest case with two classes.

Let $B = (-1,-11)/\mathbb{Q}$, with $\mathrm{i}^2 = -1$, $j^2 = -11$ and $k = \mathrm{i}j$; it is ramified exactly at 11 and ∞. Let $\mathcal{O}_1 := \mathbb{Z} + \mathbb{Z}\mathrm{i} + \mathbb{Z}\frac{1+j}{2} + \mathbb{Z}\frac{\mathrm{i}+k}{2}$. There is one other class of maximal orders, represented by an order $\mathcal{O}_2$ with $\Theta_{\mathcal{O}_2} := \sum_{x\in\mathcal{O}_2} q^{N(x)} = 1 + 6q + 6q^3 + 24q^4 + 18q^5 + \cdots$. Let $w_i := |\mathcal{O}_i^\times|/2$. Gross defines the Brandt matrices $B(m)$, $m \ge 1$, whose entries $B_{ij}(m)$ count the elements $b \in M_{ij} := I_j^{-1} I_i$ with $N(b) = m\,N(M_{ij})$, divided by $2w_j$, where $N(M_{ij})$ is the positive generator of the ideal generated by the reduced norms of elements of M_{ij}, for left ideals I_i of a fixed maximal order with right orders $\mathcal{O}_i$; he shows that $B_{ii}(m) = r_{\mathcal{O}_i}(m)/2w_i$, that $w_j B_{ij}(m) = w_i B_{ji}(m)$, and that $\sum_j B_{ij}(m) = \sigma(m)$ for $11 \nmid m$ [Gr, §§1–2]. The assignment $T_m \mapsto B(m)$ identifies the Hecke action on $M_2(\Gamma_0(11))$ with the Brandt action [Gr]. For a prime $\ell \ne 11$, $B_{ij}(\ell)$ counts the left ideals $J \subset I_i$ of the fixed order with $[I_i : J] = \ell^2$ and $J \cong I_j$; there are $\ell + 1$ such J in all, and they play the role of the ℓ-neighbours of Section 24.1.

Problem 24.4.1. $\mathcal{O}_1$ is an order, its norm form is $(a^2+ac+3c^2)+(b^2+bd+3d^2)$, and $\mathcal{O}_1^\times = \{\pm1, \pm\mathrm{i}\}$. Thus $\Theta_{\mathcal{O}_1} = \big(\sum_{a,c} q^{a^2+ac+3c^2}\big)^2$.

Eichler's mass formula $\sum_i 1/w_i = (11-1)/12$ then gives $w_2 = 3$ [Gr].

Problem 24.4.2. $B(2) = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ and $B(3) = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$, with eigenvalues $3, -2$ and $4, -1$.

Problem 24.4.3. Let $f := q\prod_{n\ge1}(1-q^n)^2(1-q^{11n})^2 = q - 2q^2 - q^3 + 2q^4 + q^5 + \cdots$. Then $\Theta_{\mathcal{O}_1} - \Theta_{\mathcal{O}_2} = -2f$, and $(1,-1)\,B(\ell) = a_\ell(f)\,(1,-1)$ for $\ell = 2, 3$.

The eigenvalues -2 and -1 are the traces of Frobenius of the elliptic curve $X_0(11)$ at 2 and 3.

24.5. A split place

Consider a theta lift at a split place whose local kernel is the indicator of $xL \subset L'$ for a matrix x; the Hecke operators are sums over edges of lattices. The following subspace count includes the local count in Problem 24.3.2 when the source has rank one.

Let Q be a power of p, and let W be the unramified extension of $\mathbb{Z}_p$ with residue field $\mathbb{F}_Q$. For lattices $L'' \subseteq L$ in a $W[1/p]$-vector space, (L'', L) is an *edge of type* i if $pL \subseteq L'' \subseteq L$ and $\dim_{\mathbb{F}_Q}(L/L'') = i$. Let $L \subset W[1/p]^N$ and $L' \subset W[1/p]^{N+g}$ be lattices, let x be an $(N+g) \times N$ matrix with $xL \subset L'$, and let ρ be the rank of x modulo p as a map $L/pL \to L'/pL'$. Here $N \geq 1$, $g \geq 0$, and $i \geq 0$ are integers, and $\begin{bmatrix} a \\ b \end{bmatrix}_Q = 0$ for $b < 0$ or $b > a$.

Problem 24.5.1. The number of edges (L_1, L') of type i with $xL \subset L_1$ is $\begin{bmatrix} N+g-\rho \\ i \end{bmatrix}_Q$. The number of edges (L, A_1) of type i with $xA_1 \subset L'$ is $\begin{bmatrix} N-\rho \\ i \end{bmatrix}_Q$.

Problem 24.5.2. For every $(N+g) \times N$ matrix x over $W[1/p]$, with $\mathbf{1}_{L,L'}(x) := [xL \subseteq L']$, one has

$$\sum_{(L_1,L') \text{ of type } i} \mathbf{1}_{L,L_1} = \sum_{l=0}^{i} Q^{(i-l)(g-l)} \begin{bmatrix} g \\ l \end{bmatrix}_Q \sum_{(L,A_1) \text{ of type } i-l} \mathbf{1}_{A_1,L'}.$$

For $N = i = 1$, $g = n-1$ and $Q = p$, this gives the count for $x \in L$ in Problem 24.3.2, with ρ the rank of the vector reduced modulo π.

Problem 24.5.3. For $0 \leq l \leq g$, the elementary symmetric polynomial e_l satisfies

$$e_l\big((Q^{(g-1)/2-s})_{0 \leq s < g}\big) = Q^{-l(g-l)/2} \begin{bmatrix} g \\ l \end{bmatrix}_Q.$$

The family of arguments is empty if $g = 0$.

Under the split-kernel hypotheses (S1)–(S3) of [Adr, §§10.3–10.4], write $Q = z^2$ for the residue-field order, with $z = p^{d/2}$ and d the even residue degree, $N_j = N_0 + jg$ with $g \geq 1$ for the ranks, and $T_i^{(j)}$ for the raw edge sums. For $0 \leq r \leq N_0$ set $\mathcal{E}_i^{(j)} = z^{-i(N_j-i)} T_i^{(j)}$ and $E_j(X) := \sum_{i=0}^{r} \mathcal{E}_i^{(j)} X^i$. For $0 \leq j < m$, the lift K_j satisfies $E_{j+1}(X) K_j = P_{j+1}(X) K_j E_j(u_{j+1} X)$ modulo X^{r+1}. If a, a' and μ are the length-weight parameters in (S3) for this lift, then $u_{j+1} = (aa')^{-1}$ is a unit, $m_{j+1} = ap^{\mu}$, and $P_{j+1}(X)$ is the product of $1 + u_{j+1} m_{j+1} \alpha X$ over the arguments α of Problem 24.5.3.

Problem 24.5.4. Suppose that $E_{j+1}(X)\, K_j = K_j\, P_{j+1}(X)\, E_j(X)$ modulo X^{r+1} for $0 \leq j < m$, with $P_{j+1}(X) = \prod_{a \in x_{j+1}} (1 + aX)$, and where $E_j(X) = \sum_i \mathcal{E}_i^{(j)} X^i$, and let Y be the concatenation, with multiplicities, of the families $x_1, \ldots, x_m$ and $\mathcal{K} := K_{m-1} \cdots K_0$. Then $\mathcal{K}\, \mathcal{E}_r^{(0)} = \sum_{l=0}^{r} (-1)^l h_l(Y)\, \mathcal{E}_{r-l}^{(m)}\, \mathcal{K}$, where h_l is the complete symmetric polynomial.

The coefficient of $\mathcal{E}_{r-l}^{(m)} \mathcal{K}$ is $(-1)^l h_l(Y)$. Problem 24.2.4 gives an analogous relation for a single orthogonal theta lift.

This chapter serves one step of the proof: at the exceptional primes below the second cutoff, Section 48.4 of Chapter 48, it transfers the split Hecke operators at the members of a full mark through the chain.

In the Lean formalization. The corresponding statements of the Lean proof [GL] are `New.Data.IsWeakChainDatum.exists_heckePolyOp_lift` for one lift and `New.Data.IsWeakChainDatum.exists_hecke_transfer` for the first $j+1$ lifts, both in `New/UpperBound/Divisibility/ValuationAtNonSplitPrimes.lean`. The full-mark bound for the restricted chain is `New.Data.IsWeakChainDatum.gaugeLe_restrictedChain_of_is` in `New/UpperBound/Divisibility/LocalDivisibility.lean`.

Hints

§24.1.

24.1.1: Compare the two indices at the intersection. For the converse consider $v \in pM \setminus pL$. For the lifting step try $v + ptw$, where $v \cdot w \not\equiv 0 \pmod p$ and $t \in \mathbb{Z}$ is to be chosen.

24.1.2: The discriminant of L/pL is $(-1)^k$ times a square. A split space of dimension $2k$ has $p^{2k-1} + p^k - p^{k-1}$ isotropic vectors, counting 0.

24.1.3: Count the orbits of pairs (L, M) with $M \in \mathcal{N}_p(L)$ under the rational orthogonal group of $L \otimes \mathbb{Q}$, weighted by the inverse order of the stabilizer of the pair, in two ways. The relation is symmetric because $L \in \mathcal{N}_p(M)$.

24.1.4: Chenevier and Lannes give the orbit sizes for the two-neighbours of $E_8 \perp E_8$: 270 and 14400 for the orbits returning to its class, and 135^2 for the orbit reaching D_{16}^+ [CL, Ch. III, §3.1]. The row sums are 32895. Use Problem 24.1.3 for the second row.

§24.2.

24.2.1: Count isotropic lines in $\bar{x}^\perp$, first finding its radical in each case.

24.2.2: Use the description of M_v in Problem 24.1.1. For the theorem, separate $x \in pL$, $x \in L \setminus pL$, and $x \notin L$.

24.2.3: $\sigma_3(9) = 757$ and $\sigma_3(4) = 73$.

24.2.4: Exchange the sums by Problem 24.1.3, then use Theorem 24.2.1. On an eigenform, $T_{p^2} = T_p^2 - p^{k-1}$.

24.2.5: $M_8(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by E_4^2, and $T_2 E_4^2 = (1 + 2^7) E_4^2$. The eigenvector is orthogonal to the constants.

§24.3.

24.3.1: Work locally. Over $\mathbb{Z}_p$, $\mathcal{O} \otimes \mathbb{Z}_p = \mathbb{Z}_p \times \mathbb{Z}_p$, and a self-dual lattice is $A \oplus A^\vee$ for a $\mathbb{Z}_p$-lattice A.

24.3.2: Use the local description of L'_H from Problem 24.3.1, one component at a time.

24.3.3: Sum Problem 24.3.2 over the vectors of norm m, using $h(\bar{\pi}y, \bar{\pi}y) = p\, h(y, y)$ and $r_{\pi L}(m) = r_L(m/p)$. For the identity on class functions, reverse a neighbour edge.

24.3.4: Compare the coefficients of q in Problem 24.3.3 for $L = I_5$, using $r_{I_5}(5) = 8424$ and the values of $r(1)$. For the second row, do the same for $L = \Lambda \perp I_1$, with $r_{\Lambda \perp I_1}(5) = 8648$; or use Problem 24.1.3 together with complex conjugation, which exchanges π- and $\bar{\pi}$-neighbours and fixes both classes.

24.3.5: Use $122880 = 2 \cdot 61440$ and $184320 = 3 \cdot 61440$.

24.3.6: Compute $a_5(g)$ from the product. Then apply Problem 24.3.3 to φ and compare with Problem 24.3.5.

§24.4.

24.4.1: Write an element as $(a + c/2) + (b + d/2)\mathrm{i} + (c/2)j + (d/2)k$.

24.4.2: Find the diagonal entries from $r_{\mathcal{O}_1}(m)$ and from $\Theta_{\mathcal{O}_2}$, and the others from the row sums. Check $w_2 B_{12} = w_1 B_{21}$.

24.4.3: Both sides lie in $M_2(\Gamma_0(11))$, which has dimension 2; compare the coefficients of q^0, q^1, q^2.

§24.5.

24.5.1: Reduce the two edges modulo p and use the image and kernel of $\bar{x}$, respectively.

24.5.2: Problem 24.5.1 and the q-Vandermonde identity [Andr, Ch. 3]

$$\begin{bmatrix} a+b \\ i \end{bmatrix}_Q = \sum_{l} Q^{(i-l)(b-l)} \begin{bmatrix} b \\ l \end{bmatrix}_Q \begin{bmatrix} a \\ i-l \end{bmatrix}_Q .$$

24.5.3: Use the Gaussian binomial theorem

$$\prod_{s=0}^{g-1} (1 + Q^s z) = \sum_{l} Q^{l(l-1)/2} \begin{bmatrix} g \\ l \end{bmatrix}_Q z^l .$$

24.5.4: Compose the identities in increasing stage order and invert the resulting scalar series.

References

[Adr] J. Adriazola, *The Alpöge proof of the Riemann hypothesis*, manuscript, revised 24 November 2026, `https://github.com/jimjamzola/RH-Alpoge-Reconstruction`.

[An] A. N. Andrianov, *Quadratic Forms and Hecke Operators*, Grundlehren Math. Wiss. 286, Springer, 1987.

[Andr] G. E. Andrews, *The Theory of Partitions*, Encyclopedia Math. Appl. 2, Addison-Wesley, 1976.

[CL] G. Chenevier and J. Lannes, *Automorphic Forms and Even Unimodular Lattices*, Ergeb. Math. Grenzgeb. (3) 69, Springer, 2019.

[CoS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Grundlehren Math. Wiss. 290, Springer, 1999.

[Ei52] M. Eichler, *Quadratische Formen und orthogonale Gruppen*, Grundlehren Math. Wiss. 63, Springer, 1952.

[Ei73] M. Eichler, *The basis problem for modular forms and the traces of the Hecke operators*, in *Modular Functions of One Variable I*, Lecture Notes in Math. 320, Springer, 1973, 75–151.

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[Gr] B. H. Gross, *Heights and the special values of L-series*, in *Number Theory (Montreal, 1985)*, CMS Conf. Proc. 7, Amer. Math. Soc., 1987, 115–187.

[Iw] H. Iwaniec, *Topics in Classical Automorphic Forms*, Grad. Stud. Math. 17, Amer. Math. Soc., 1997.

[Kn] M. Kneser, *Klassenzahlen definiter quadratischer Formen*, Arch. Math. **8** (1957), 241–250.

[Pi] A. Pizer, *An algorithm for computing modular forms on* $\Gamma_0(N)$, J. Algebra **64** (1980), 340–390.

[Ra] S. Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104** (1982), 469–515.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. 7, Springer, 1973.