THE SIEGEL–WEIL FORMULA IN THE SMALLEST CASE

Prerequisites. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$, theta series of even unimodular lattices, and the mass formula, as in [SeA, Ch. V, §2.3 and Ch. VII, §§2–6]. Quadratic forms over finite fields [SeA, Ch. IV, §1.7] and Hensel's lemma [SeA, Ch. II, §2.2]. For §22.3: the space of modular forms of weight n on $\Gamma_0(4)$ with character χ^n, its dimension and its basis in terms of θ [Ko, Ch. IV, §1], and its Eisenstein series [DS, §4.5]. From Chapter 21, *Masses and the count of forms*: the class sets Cl_n, the groups Γ_L, the masses M_n, Problems 21.1.2, 21.1.4 and 21.2.3, and the forms of §21.3 with Problem 21.3.2.

For a positive definite lattice, the isometry classes in its genus form a finite set. In the convergent range, Siegel's theorem identifies the average of their theta series, with weights proportional to the reciprocals of the automorphism orders, as an Eisenstein series. We check this average for even unimodular lattices and for the rank-five Hermitian class set of Chapter 21. We then express a rank-one lift norm in terms of averaged representation numbers.

22.1. Even unimodular lattices

Let L be an even unimodular positive definite lattice of rank $m > 0$, with inner product $x \cdot y$, and let $r_L(j)$ be the number of $x \in L$ with $x \cdot x = 2j$. Then $8 \mid m$, and $\theta_L(\tau) := \sum_{j \geq 0} r_L(j) q^j$, $q = e^{2\pi i \tau}$, is a modular form of weight $k := m/2$ for $\mathrm{SL}_2(\mathbb{Z})$ [SeA, Ch. VII, Theorem 8]. Let $\mathcal{M}_k$ be the space of such forms and

$$E_k := 1 - \frac{2k}{B_k} \sum_{j \geq 1} \sigma_{k-1}(j) q^j, \qquad \Delta := q \prod_{j \geq 1} (1 - q^j)^{24} = \sum_{j \geq 1} \tau(j) q^j,$$

so that $E_4 = 1 + 240 \sum \sigma_3(j) q^j$, $E_8 = 1 + 480 \sum \sigma_7(j) q^j$, $E_{12} = 1 + \frac{65520}{691} \sum \sigma_{11}(j) q^j$. (Serre writes E_2, E_4, E_6 for these, indexing by half the weight.) The spaces $\mathcal{M}_4$ and $\mathcal{M}_8$ are spanned by E_4 and E_8, and $\mathcal{M}_{12}$ by E_{12} and Δ [SeA, Ch. VII, §3.2]. We write E_8 also for the root lattice; the context decides. Let $D_{16}^+ := D_{16} \cup (D_{16} + (\frac{1}{2}, \ldots, \frac{1}{2}))$, where $D_{16} = \{x \in \mathbb{Z}^{16} : \sum x_i \text{ even}\}$ [CoS, Ch. 4, §7].

Problem 22.1.1. $r_{E_8}(j) = 240\, \sigma_3(j)$ for $j \geq 1$. In particular E_8 has 240 roots and 2160 vectors with $x \cdot x = 4$.

Problem 22.1.2. $\theta_{E_8 \oplus E_8} = \theta_{D_{16}^+} = E_8$. Both lattices have 480 roots and 61920 vectors with $x \cdot x = 4$, and $\sigma_7(j) = \sigma_3(j) + 120 \sum_{i=1}^{j-1} \sigma_3(i) \sigma_3(j-i)$ for $j \geq 1$.

Problem 22.1.3. $E_8 \oplus E_8$ and D_{16}^+ are not isometric.

So the theta series does not determine the lattice. Milnor observed that the tori $\mathbb{R}^{16}/(E_8 \oplus E_8)$ and $\mathbb{R}^{16}/D_{16}^+$ therefore have the same Laplace spectrum without being isometric [Mi].

Problem 22.1.4. If Λ_{24} is an even unimodular lattice of rank 24 without roots (the Leech lattice [CoS, Ch. 4, §11]), then $\theta_{\Lambda_{24}} = E_{12} - \frac{65520}{691} \Delta$, and Λ_{24} has 196560 vectors with $x \cdot x = 4$.

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In rank 24 write $\theta_L = E_{12} + c_L\Delta$ [SeA, Ch. VII, §6.6]. Let C_m be the set of isometry classes of positive definite even unimodular lattices of rank m, let $g_L := |O(L)|$, and let $\mu_m := \sum_{L\in C_m} g_L^{-1}$ be the mass. Siegel proved [Si] that

$$\sum_{L\in C_m} \frac{\theta_L}{g_L} = \mu_m\, E_{m/2}, \tag{1}$$

see [SeA, Ch. VII, §6.5, Remark], and the Minkowski–Siegel mass formula [SeA, Ch. V, §2.3]

$$\mu_m = \frac{|B_k|}{2k} \prod_{j=1}^{k-1} \frac{|B_{2j}|}{4j}, \qquad m = 2k.$$

One has $C_8 = \{E_8\}$ with $g_{E_8} = |W(E_8)| = 696729600$, and $C_{16} = \{E_8 \oplus E_8, D_{16}^+\}$ with $g = 2\,|W(E_8)|^2$ and $g = 2^{15}\,16!$ respectively [SeA, Ch. V, §2.3].

Problem 22.1.5. The mass formula holds for $m = 8$ and $m = 16$. In the average obtained by dividing (1) by μ_{16} for $m = 16$, $E_8 \oplus E_8$ has weight $286/691$ and D_{16}^+ has weight $405/691$.

Problems 22.1.1 and 22.1.2 give (1) for ranks 8 and 16. In rank 24 the individual theta series can differ.

Problem 22.1.6. For $m = 24$, (1) is equivalent to $\sum_{L\in C_{24}} c_L/g_L = 0$. Hence the mass-weighted mean number of roots of an even unimodular lattice of rank 24 is $65520/691$; in particular $\mu_{24}^{-1}\sum_L \theta_L/g_L$ is not the theta series of a lattice.

22.2. Local densities

Why should an average of theta series be an Eisenstein series? Its coefficients are averages of representation numbers, and Siegel's theorem says that these are products of local quantities. Let $Q(x) := x\cdot x/2$ on L. For a prime p and $j \geq 1$ the *local density* is

$$\alpha_p(j) := \lim_{r\to\infty} p^{r(1-m)}\, \#\{x \in L/p^rL : Q(x) \equiv j \pmod{p^r}\},$$

and the *archimedean density* is $\alpha_\infty(j) := \frac{d}{dt}\,\mathrm{vol}\{x \in L\otimes\mathbb{R} : Q(x) \leq t\}\big|_{t=j}$, the volume being normalized so that L has covolume 1. In this language Siegel's theorem reads

$$\frac{1}{\mu_m} \sum_{L\in C_m} \frac{r_L(j)}{g_L} = \alpha_\infty(j) \prod_p \alpha_p(j) \qquad (j \geq 1), \tag{2}$$

and the densities depend only on $L\otimes\mathbb{Z}_p$ and $L\otimes\mathbb{R}$, which are the same for every $L \in C_m$. For every prime p, the reduction L/pL with the form Q mod p is isometric to the orthogonal sum H^k of $k = m/2$ hyperbolic planes, $Q = x_1y_1 + \cdots + x_ky_k$ [Ca, Ch. 8].

Problem 22.2.1. $\alpha_\infty(j) = (2\pi)^k j^{k-1}/\Gamma(k)$.

Problem 22.2.2. For $j \in \mathbb{F}_p$, the number of $(x,y) \in \mathbb{F}_p^k \times \mathbb{F}_p^k$ with $x_1y_1 + \cdots + x_ky_k = j$ is $p^{2k-1} - p^{k-1}$ for $j \neq 0$, and $p^{2k-1} + p^k - p^{k-1}$ for $j = 0$.

Problem 22.2.3. If $p \nmid j$, then $\alpha_p(j) = 1 - p^{-k}$.

Problem 22.2.4. If $p\|j$, then $\alpha_p(j) = (1-p^{-k})(1+p^{1-k})$. More generally, for $v = v_p(j)$, one has $\alpha_p(j) = (1-p^{-k})\sum_{i=0}^{v} p^{i(1-k)}$.

Problem 22.2.5. $\alpha_\infty(j)\prod_p \alpha_p(j) = -\frac{2k}{B_k}\,\sigma_{k-1}(j)$.

Thus the right side of (2) is the jth coefficient of E_k, and (2) is (1) coefficient by coefficient. With Problems 22.1.1 and 22.1.2 this proves the following.

Theorem 22.2.1. *For $m = 8$ and $m = 16$, and every $j \geq 1$, the mass-weighted mean of $r_L(j)$ over C_m is $\alpha_\infty(j)\prod_p \alpha_p(j)$.*

Weil reformulated (1) as an identity on adelic groups: the integral of the theta function over $O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})$ taken with the invariant measure of total mass one, is an Eisenstein series, under hypotheses that include the absolute convergence of the Eisenstein series [We65, Theorem 5]. For a positive definite space the quotient is compact. With the theta kernel attached to a lattice, the integral with measure of total mass one is the normalized genus average. This is the *Siegel–Weil formula in the convergent range.*

22.3. The Hermitian class set over $\mathbb{Z}[\mathrm{i}]$

Return to the class sets Cl_n of Hermitian lattices over $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$ from Chapter 21. For $L \in \mathrm{Cl}_n$ let $\theta_L(\tau) := \sum_{x\in L} q^{h(x,x)}$, and let $\theta := \sum_{j\in\mathbb{Z}} q^{j^2}$. Let χ be the nontrivial character modulo 4, and $\mathcal{M}_n(4,\chi^n)$ the space of modular forms of weight n on $\Gamma_0(4)$ with character χ^n. It has dimension $1 + \lfloor n/2\rfloor$ and basis $\theta^{2n-4i}F^i$, $0 \leq i \leq n/2$, where $F := \sum_{\substack{j\geq 1\\ j \text{ odd}}} \sigma_1(j)q^j$ [Ko, Ch. IV, §1]; so an element is determined by its coefficients of $q^0, \ldots, q^{\lfloor n/2\rfloor}$. Each θ_L lies in $\mathcal{M}_n(4,\chi^n)$.

For Hermitian lattices Siegel's theorem is due to Braun [Br], and it is a case of Weil's theorem [We65, Theorem 5]; for the unitary formulation see [Li, Theorem 2.3.1]. It reads, for $n \geq 3$,

$$\sum_{L\in\mathrm{Cl}_n} \frac{\theta_L}{|\Gamma_L|} = M_n\, E^{(n)}, \qquad E^{(n)} = 1 + \sum_{j\geq 1} \alpha_\infty(j)\prod_p \alpha_p(j)\, q^j, \tag{3}$$

with the densities of I_n defined as in §22.2 on the underlying $\mathbb{Z}$-lattice of rank $2n$, with $Q(x) := h(x,x)$, and $E^{(n)}$ an Eisenstein series in $\mathcal{M}_n(4,\chi^n)$. The condition $n \geq 3$ is where Eisenstein series of weight n converge absolutely.

Problem 22.3.1. For τ in the upper half-plane and $s \in \mathbb{R}$, the sum $\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} |c\tau + d|^{-s}$ converges if and only if $s > 2$.

Problem 22.3.2. $\theta_{I_n} = \theta^{2n}$, and $\theta_{\Lambda\perp I_1}(\tau) = E_4(2\tau)\,\theta(\tau)^2$, with Λ as in Problem 21.2.3.

For $n = 3$ and 4, $\mathrm{Cl}_n = \{I_n\}$ (Problem 21.1.4), so (3) says that θ^{2n} is itself an Eisenstein series. This is Jacobi's formula for sums of $2n$ squares; for $n = 4$ and positive odd j it reads $r_8(j) = 16\,\sigma_3(j)$. Here are its local factors.

Problem 22.3.3. For I_4 and positive odd j, $\alpha_\infty(j) = \pi^4 j^3/6$ and $\alpha_\infty(j)\prod_{p \text{ odd}} \alpha_p(j) = 16\,\sigma_3(j)$.

Problem 22.3.4. For positive odd j, the number of $x \in (\mathbb{Z}/8\mathbb{Z})^8$ with $x_1^2 + \cdots + x_8^2 \equiv j \pmod 8$ is 8^7. For every $r \geq 3$ the number of solutions modulo 2^r is 2^{7r}. Deduce that $\alpha_2(j) = 1$ and that the full density product is $16\,\sigma_3(j)$.

In rank 5 the class set has two elements, I_5 and $\Lambda \perp I_1$ (Problem 21.2.3), and $\dim \mathcal{M}_5(4,\chi) = 3$, and its Eisenstein subspace has dimension 2. Let $e_4(j) := \sum_{d\mid j,\, d \text{ odd}} \chi(d)\, d^4$ and $e_4'(j) := \sum_{d\mid j,\, j/d \text{ odd}} \chi(j/d)\, d^4$. The series $G_5 := 1 + \frac{4}{5}\sum_{j\ge 1}(e_4(j) + 16e_4'(j))q^j$ is a combination of the two Eisenstein series of $\mathcal{M}_5(4,\chi)$ [DS, §4.5].

Problem 22.3.5. In Cl_5, the weights $|\Gamma_L|^{-1}/M_5$ are $3/5$ for I_5 and $2/5$ for $\Lambda \perp I_1$.

Problem 22.3.6. $\frac{3}{5}\,\theta^{10} + \frac{2}{5}\,E_4(2\tau)\theta^2 = G_5$. The coefficients of q in θ^{10}, $E_4(2\tau)\theta^2$ and G_5 are 20, 4 and $68/5$.

In rank 5 the mass-weighted average is therefore Eisenstein. To identify the cusp parts of the individual theta series, consider the following form. Let $f := \frac{1}{4}\sum_{\alpha\in\mathcal{O}} \alpha^4 q^{|\alpha|^2} = q - 4q^2 + 16q^4 - 14q^5 + \cdots$, the theta series of $\mathcal{O}$ with the harmonic polynomial α^4; it is a cusp form in $\mathcal{M}_5(4,\chi)$ [Iw, Ch. 10].

Problem 22.3.7. $\theta^{10} - E_4(2\tau)\theta^2 = 16f$. Hence $\theta^{10} = G_5 + \frac{32}{5}f$ and $E_4(2\tau)\theta^2 = G_5 - \frac{48}{5}f$.

The identity for θ^{10} in Problem 22.3.7 is the theta-series form of Liouville's ten-square formula [Liouville], $r_{10}(j) = \frac{4}{5}(e_4(j) + 16e_4'(j)) + \frac{8}{5}\sum_{|\alpha|^2=j}\alpha^4$. The cusp parts $\frac{32}{5}f$ and $-\frac{48}{5}f$ cancel in the average with weights $3/5$ and $2/5$, and only there. Since $f \neq 0$, neither individual theta series is Eisenstein.

22.4. Gram rows and nonvanishing

In the definite model of Chapter 21 the theta lifts go from forms on Cl_n to forms on $\mathrm{Cl}_{n'}$. Their Gram matrices are sums over the target class set, which can be large even when its mass is explicit (Chapter 21, §21.2). We take $n = 1$.

Let $n' \ge 1$. For an integer $a \ge 0$ and $L' \in \mathrm{Cl}_{n'}$ let

$$v_a(L') := \sum_{x\in L'} x^a\, e^{-\pi h(x,x)} \in \mathrm{Sym}^a(L'_{\mathbb{C}}),$$

with x^a as in §21.3. For $4 \mid a$, let $\varphi(I_1)$ be the standard basis vector of $\mathrm{Sym}^a(\mathbb{C})$. The lift of the next chapter, *Definite theta lifts and their norms*, satisfies $v_a = 4\Theta_{(a)}\varphi$, since $|\Gamma_{I_1}| = 4$ (Problem 21.3.3). For $x, y \in L'$ let $\mathrm{Gram}(x,y)$ be the matrix with entries $h(x,x)$, $h(x,y)$, $h(y,x)$, $h(y,y)$, and for a 2×2 Hermitian matrix T over $\mathcal{O}$ put

$$R_{n'}(T) := \sum_{L'\in \mathrm{Cl}_{n'}} \frac{1}{|\Gamma_{L'}|}\, \#\{(x,y) \in L'^2 : \mathrm{Gram}(x,y) = T\}.$$

Problem 22.4.1. v_a is a form of weight (a) on $\mathrm{Cl}_{n'}$, and $v_a = 0$ unless $4 \mid a$.

Problem 22.4.2. $\|v_a\|^2 = \sum_T R_{n'}(T)\, T_{12}^a\, e^{-\pi\,\mathrm{tr}\,T}$, the sum running over positive semi-definite T.

For this rank-one source the norm depends on the target genus through the averaged counts $R_{n'}(T)$. For $n' > 4$ the degree-two Siegel–Weil formula gives, for each positive definite integral Hermitian T,

$$R_{n'}(T) = M_{n'}\, \alpha_\infty(T) \prod_p \alpha_p(T).$$

Here $\alpha_p(T)$ is the limit, as $r \to \infty$, of $p^{r(4-4n')}$ times the number of pairs in $(I_{n'}/p^r I_{n'})^2$ with Gram matrix congruent to T modulo p^r. At infinity take the density of the image of Lebesgue measure under the Gram map, with covolume one for $I_{n'}^2$ and for the lattice of integral Hermitian matrices in the target [We65, Theorem 5]. The condition $n' > 4$ is the range of absolute convergence of the Eisenstein series for pairs. If a positive definite T is represented over every $\mathbb{Z}_p$, then $R_{n'}(T) > 0$. Some lattice in the genus therefore represents T, although a prescribed lattice can fail to do so.

Problem 22.4.3. Let 1_2 be the identity matrix. Then I_5 contains 320 pairs (x, y) with $\mathrm{Gram}(x, y) = 1_2$, and $\Lambda \perp I_1$ contains none. Hence $R_5(1_2) = 1/384$.

For $a = 0$ one can take the lift one step further and pair it with the constant form.

Problem 22.4.4. For $n' \geq 3$, $\langle v_0, 1\rangle = M_{n'}\, E^{(n')}(\mathrm{i}/2)$. For $n' = 5$ this is $G_5(\mathrm{i}/2)/73728$, and $G_5(\mathrm{i}/2) = 2.06195\ldots$.

The classical Rallis inner product formula gives a nonvanishing criterion. Let φ be a nonzero form on Cl_n that is a decomposable vector in an irreducible automorphic representation, and form its theta lift $\Theta\varphi$ to $\mathrm{Cl}_{n'}$, $n' > 4n$, with decomposable test data. In the doubling calculation of $\|\Theta\varphi\|^2$, one integrates the doubled theta kernel over the target unitary quotient. For the lattice data at the standard stabilizer level this inner integral is a genus average, as in Problem 22.4.2. The Siegel–Weil formula evaluates the theta integral [Ra, JSL]. With the standard doubling normalization, the ratio $\|\Theta\varphi\|^2/\|\varphi\|^2$ is, up to a nonzero measure constant, a partial twisted standard L-value divided by the unramified normalizing product of Tate L-values, times the local integrals at the remaining places [Li, §2.4]. In this range the Euler products converge absolutely and are nonzero, so nonvanishing of those local integrals implies nonvanishing of the lift. For sums of decomposable data or vectors, expand the pairing by sesquilinearity.

The genus averages in this chapter illustrate two steps of the proof: the estimate for the main term of the second moment (Chapter 36) and the norms of the single lifts of the chain (Chapter 55).

In the Lean formalization. The corresponding statements of the Lean proof [GL] are `IsFillingData.mul_sum_norm_sum_cosetAmplitude_sq_le` in `grh/src/New/Spanning/MainTerm/EnergyEsti` for the main term, `exists_adjoint_singleLift_comp_eq` in `grh/src/New/Determinant/Norms/AdjointThetal` for the single lifts, and `inner_product_formula` in `grh/src/Lib/SiegelWeil/RallisInner/Doubling.lean` for the unfolding by doubling.

Hints

§22.1.

- **22.1.1:** $\dim \mathcal{M}_4 = 1$.
- **22.1.2:** $\dim \mathcal{M}_8 = 1$, and $\theta_{E_8 \oplus E_8} = \theta_{E_8}^2$. For the roots of D_{16}^+, compute $x \cdot x$ on the non-trivial coset.
- **22.1.3:** Is each lattice spanned by its roots?
- **22.1.4:** Write $\theta_{\Lambda_{24}} = E_{12} + c\Delta$ and compare coefficients of q; then use $\tau(2) = -24$.
- **22.1.5:** $B_2, B_4, \ldots, B_{16}$; for $m = 16$ one needs $B_8 = -1/30$ and $B_{14} = 7/6$.
- **22.1.6:** Subtract $\mu_{24} E_{12}$ from both sides of (1), and compare coefficients of q.

§22.2.

22.2.1: Since L is unimodular, the volume is that of a Euclidean ball of radius $\sqrt{2t}$ in dimension $2k$.

22.2.2: Induction on k: count first the solutions with $x_k = 0$, then those with $x_k \neq 0$.

22.2.3: Every solution modulo p of $Q(x) \equiv j$ has $x \not\equiv 0$, and the gradient of Q is nonzero there, for every p, because L is unimodular; apply Hensel's lemma.

22.2.4: Separate the solutions with $x \not\equiv 0 \pmod p$, which lift as in Problem 22.2.3, from those with $x \equiv 0$, for which $Q(x) \equiv 0 \pmod{p^2}$. When $p^2 \mid j$, what density comes from vectors divisible by p?

22.2.5: The Euler product of $\zeta(k)$, Euler's formula for $\zeta(k)$ in terms of B_k, and $\sigma_{k-1}(j) = j^{k-1}\sigma_{1-k}(j)$.

§22.3.

22.3.1: Count the pairs with $N \le |c\tau + d| < N + 1$.

22.3.2: $h(x, x)$ is additive on orthogonal sums, and $h(x, x) = x \cdot x$ is even on Λ.

22.3.3: As in Problem 22.2.1, but $\{Q \le t\}$ is now the ball of radius $\sqrt{t}$. The counting argument of Problems 22.2.3–22.2.4 uses only that $Q \bmod p$ is isometric to H^4, which holds for $x_1^2 + \cdots + x_8^2$ and odd p. Then $\zeta(4) = \pi^4/90$.

22.3.4: Write the count as $\frac{1}{8}\sum_{t \bmod 8} e(-tj/8)\, g(t)^8$ with the Gauss sums $g(t) := \sum_{x \bmod 8} e(tx^2/8)$, where $e(u) := \exp(2\pi i u)$. For $r \ge 3$, how does adding 2^{r-1} to the first odd coordinate change the norm modulo 2^{r+1}?

22.3.5: Problem 21.1.2 and $|\Gamma_\Lambda| = 46080$.

22.3.6: Both sides lie in $\mathcal{M}_5(4, \chi)$; compare the coefficients of q^0, q^1, q^2.

22.3.7: As in Problem 22.3.6; then combine with it.

§22.4.

22.4.1: Substitute $x \mapsto \gamma x$ for $\gamma \in \Gamma_{L'}$, and $x \mapsto \mathrm{i}x$.

22.4.2: Problem 21.3.2: $\langle x^a, y^a \rangle = h(x, y)^a$. Then group the pairs by their Gram matrix.

22.4.3: Problem 21.2.3: the vectors of $\Lambda \perp I_1$ with $h(x, x) = 1$ are ue, $u \in \mu_4$.

22.4.4: What is $v_0(L')$ as a value of $\theta_{L'}$?

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