CONGRUENCE MODULES

Prerequisites. Modules over a Dedekind domain and the elementary divisor theorem. Ideals in quadratic fields. Modular forms of level 1: the structure of M_k and S_k [SeA, Ch. VII, §3.2, Thm. 4], Eisenstein series and Δ [SeA, Ch. VII, §§4.2–4.3], Hecke operators and their action on q-expansions, and normalized eigenforms [SeA, Ch. VII, §§5.3–5.5]; Serre writes the weight as $2k$ and numbers the Bernoulli numbers differently. The Petersson product, for which the Hecke operators are self-adjoint [DS, §5.5]. From Chapter 26, *Denominators and the Stickelberger element*: the cost of a denominator (Problem 26.1.5) and Theorem 26.2.2.

In Chapter 26 a linear map that does not respect integral structures cost $\log[M + T(L) : M]$. One step of the proof asks for an operator clearing the denominators of an orthogonal projection onto the image of a theta lift restricted to good forms. It tests that operator on a restricted set of integral forms. Projection onto a sum of whole Hecke eigenspaces is a model for this problem. What does such a projection cost, and what controls the cost? On the full lattice the answer is a congruence module. We compute it for level 1, in weight 12, where it is $\mathbb{Z}/691\mathbb{Z}$, and in weight 24, where it detects a congruence modulo the unique prime ideal above 144169 between the two newforms.

29.1. Congruence modules

Let $\mathcal{O}$ be a Dedekind domain with fraction field K. A *lattice* in a finite-dimensional K-vector space V is a finitely generated $\mathcal{O}$-submodule that spans V. Let $V = V_1 \oplus V_2$, let π_1, π_2 be the two projections, and let $\Lambda \subset V$ be a lattice. The *congruence module* of V_1 in Λ is

$$\mathrm{C}(\Lambda; V_1) := \pi_1(\Lambda)/(\Lambda \cap V_1).$$

Problem 29.1.1. $\pi_1(\Lambda) = V_1 \cap (\Lambda + V_2)$, and

$$\mathrm{C}(\Lambda; V_1) \cong \Lambda/\big((\Lambda \cap V_1) \oplus (\Lambda \cap V_2)\big) \cong \mathrm{C}(\Lambda; V_2).$$

In particular $\mathrm{C}(\Lambda; V_1)$ is a finitely generated torsion $\mathcal{O}$-module, and it vanishes exactly when $\Lambda = (\Lambda \cap V_1) \oplus (\Lambda \cap V_2)$.

Problem 29.1.2. $(\Lambda + \pi_1(\Lambda))/\Lambda \cong \mathrm{C}(\Lambda; V_1)$. For $\mathcal{O} = \mathbb{Z}$, the least positive integer c with $c\,\pi_1(\Lambda) \subseteq \Lambda$ is the exponent of $\mathrm{C}(\Lambda; V_1)$.

So for $\mathcal{O} = \mathbb{Z}$ the index $[\Lambda + \pi_1(\Lambda) : \Lambda]$ is $|\mathrm{C}(\Lambda; V_1)|$: measured like the index cost of Problem 26.1.5 (stated there for injective maps), the projection costs $\log|\mathrm{C}(\Lambda; V_1)|$.

Problem 29.1.3. Let $N \geq 1$, $V = \mathbb{Q}^2 = \mathbb{Q}e_1 \oplus \mathbb{Q}e_2$, and $\Lambda = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod N\}$. Then $\mathrm{C}(\Lambda; \mathbb{Q}e_1) \cong \mathbb{Z}/N\mathbb{Z}$, whereas $\mathrm{C}(\mathbb{Z}^2; \mathbb{Q}e_1) = 0$.

The lattices of interest carry an action of a Hecke algebra. Let X be a finite set, and let $\mathbf{T} \subseteq \prod_{x \in X} \mathcal{O}$ be an $\mathcal{O}$-subalgebra, finitely generated as an $\mathcal{O}$-module, which separates the points of X. Write $t(x)$ for the coordinates of $t \in \mathbf{T}$. Let $\mathbf{T}$

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act on $V = \bigoplus_{x\in X} V_x$ by $t \cdot v = (t(x)v_x)_x$, and let Λ be a $\mathbf{T}$-stable lattice. The *congruence ideal* of x is

$$\mathfrak{c}_x := \{t(x) : t \in \mathbf{T},\ t(y) = 0 \text{ for all } y \neq x\} \subseteq \mathcal{O},$$

and we write $\mathrm{C}(\Lambda; V_x)$ for the congruence module with respect to $V = V_x \oplus \bigoplus_{y\neq x} V_y$.

Problem 29.1.4. $\mathfrak{c}_x$ annihilates $\mathrm{C}(\Lambda; V_x)$. If $\mathcal{O}$ is a discrete valuation ring with normalized valuation v, then $\mathrm{len}_{\mathcal{O}} \mathrm{C}(\Lambda; V_x) \leq \dim V_x \cdot \min v(\mathfrak{c}_x)$.

Problem 29.1.5. If $\Lambda \cong \mathbf{T}$ as a $\mathbf{T}$-module, then $\mathrm{C}(\Lambda; V_x) \cong \mathcal{O}/\mathfrak{c}_x$. For $\mathbf{T} = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod N\}$ acting on itself, $\mathfrak{c}_x = N\mathbb{Z}$ at both points.

Thus $\mathfrak{c}_x$ annihilates the obstruction to integral projection onto V_x. When the lattice is free of rank one over $\mathbf{T}$, this obstruction module is isomorphic to $\mathcal{O}/\mathfrak{c}_x$.

A congruence module can also be read off from a pairing. Suppose that Λ carries a *perfect* pairing $\langle\,,\rangle\colon \Lambda \times \Lambda \to \mathcal{O}$, that is, an $\mathcal{O}$-bilinear form for which $\lambda \mapsto \langle\lambda, \cdot\rangle$ is an isomorphism $\Lambda \to \mathrm{Hom}_{\mathcal{O}}(\Lambda, \mathcal{O})$, and that $\langle V_1, V_2\rangle = \langle V_2, V_1\rangle = 0$. For a lattice $L \subset V_1$ let $L^\vee := \{v \in V_1 : \langle v, L\rangle \subseteq \mathcal{O}\}$.

Problem 29.1.6. $\pi_1(\Lambda) = (\Lambda \cap V_1)^\vee$. Hence $\mathrm{C}(\Lambda; V_1) \cong (\Lambda \cap V_1)^\vee/(\Lambda \cap V_1)$.

Problem 29.1.7. In the setting of Problem 29.1.6: (i) if f is a basis of $\Lambda \cap V_1$ over $\mathcal{O}$, then $\mathrm{C}(\Lambda; V_1) \cong \mathcal{O}/\langle f, f\rangle\mathcal{O}$; (ii) if x, y form an $\mathcal{O}$-basis of $\Lambda \cap V_1$ with $\langle x, x\rangle = \langle y, y\rangle = 0$, then $\mathrm{C}(\Lambda; V_1) \cong (\mathcal{O}/\langle x, y\rangle\mathcal{O}) \oplus (\mathcal{O}/\langle y, x\rangle\mathcal{O})$. For $\Lambda = \mathbb{Z}^2$ with the dot product and $V_1 = \mathbb{Q}(1, 1)$, $V_2 = \mathbb{Q}(1, -1)$, the congruence module is $\mathbb{Z}/2\mathbb{Z}$.

For a self-dual lattice the congruence module of an orthogonal summand is its discriminant module, the quotient in Problem 29.1.6. In Section 29.4 the discriminant of the cohomological eigenlattice detects the primes in the support of the eigenform's congruence module in the range of Hida's theorem. Compare [DDT, §4.4, Lemma 4.17], where a power of the congruence ideal of the Hecke algebra is shown to lie in the discriminant ideal of the eigenform's piece of a self-dual lattice, with equality when the lattice is free over the Hecke algebra.

29.2. The Eisenstein line in weight 12

For even $k \geq 4$ let M_k and S_k be the spaces of modular forms and cusp forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, identified with their q-expansions, and let $M_k(\mathbb{Z}) := M_k \cap \mathbb{Z}[[q]]$ and $S_k(\mathbb{Z}) := S_k \cap \mathbb{Z}[[q]]$. Let $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$ and

$$E_k := 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n, \qquad \Delta := q \prod_{n\geq 1} (1 - q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n,$$

so that $E_4 = 1 + 240 \sum \sigma_3(n) q^n$, $E_6 = 1 - 504 \sum \sigma_5(n) q^n$, and $B_{12} = -691/2730$ [SeA, Ch. VII, §4]. Write $M_k(\mathbb{Q}) := M_k \cap \mathbb{Q}[[q]]$ and $S_k(\mathbb{Q}) := S_k \cap \mathbb{Q}[[q]]$. We use the decomposition $M_k(\mathbb{Q}) = \mathbb{Q}E_k \oplus S_k(\mathbb{Q})$, in which both summands are stable under the Hecke operators.

Problem 29.2.1. $M_{12}(\mathbb{Z}) = \mathbb{Z}E_4^3 \oplus \mathbb{Z}\Delta$.

Problem 29.2.2. $E_4^3 = E_{12} + \frac{432000}{691}\Delta$. Hence $\pi_\Delta(M_{12}(\mathbb{Z})) = \frac{1}{691}\mathbb{Z}\Delta$, and $\mathrm{C}(M_{12}(\mathbb{Z}); \mathbb{Q}\Delta) \cong \mathbb{Z}/691\mathbb{Z}$.

The same computation for the Eisenstein line works in every weight.

Problem 29.2.3. Let $k \geq 4$ be even, and let N_k be the numerator of $B_k/2k$ in lowest terms. Then $\mathrm{C}(M_k(\mathbb{Z}); \mathbb{Q}E_k) \cong \mathbb{Z}/N_k\mathbb{Z}$. For $k = 4, 6, 8, 10, 14$ one has $N_k = \pm 1$, and $|N_{16}| = 3617$, $|N_{22}| = 77683 = 131{\cdot}593$, $|N_{24}| = 236364091 = 103{\cdot}2294797$.

Let $\mathbf{T}_{12}$ be the $\mathbb{Z}$-algebra of endomorphisms of $M_{12}(\mathbb{Q})$ generated by the Hecke operators T_n, $n \geq 1$, normalized so that $T_n\Delta = \tau(n)\Delta$ and $T_nE_{12} = \sigma_{11}(n)E_{12}$ [SeA, Ch. VII, §5]. Map $\mathbf{T}_{12}$ to $\mathbb{Z} \times \mathbb{Z}$ by its eigenvalues on (Δ, E_{12}).

Problem 29.2.4. Each T_n preserves $M_{12}(\mathbb{Z})$, and $\mathbf{T}_{12} = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod{691}\}$. Thus the congruence ideal of Δ is $691\mathbb{Z}$.

After localization at 691, the length bound of Problem 29.1.4 is attained. Consider now the global $\mathbf{T}_{12}$-module.

Problem 29.2.5. $M_{12}(\mathbb{Z})$ is not free of rank one over $\mathbf{T}_{12}$.

The prime 691 is also the one in Theorem 26.2.2: since $691 \mid B_{12}$, the eigenspace $A^{(679)}$ of the 691-part of the class group of $\mathbb{Q}(\zeta_{691})$ is nonzero. That the same prime measures both is not an accident; Chapter 30 builds the class from the congruence.

29.3. Two newforms of weight 24

The space S_{24} has dimension 2 [SeA, Ch. VII, §3.2], and its two normalized eigenforms are not rational. Let $D := 144169$, a prime with $D \equiv 1 \pmod{24}$, let $K := \mathbb{Q}(\sqrt{D})$, $\mathcal{O} := \mathcal{O}_K$, and $\mathfrak{p} := \sqrt{D}\,\mathcal{O}$.

Problem 29.3.1. $S_{24}(\mathbb{Z}) = \mathbb{Z}g_1 \oplus \mathbb{Z}g_2$, where $g_1 = \Delta E_4^3 - 696\Delta^2 = q + 195660q^3 + \cdots$ and $g_2 = \Delta^2 = q^2 - 48q^3 + \cdots$.

On S_{24} one has $a_n(T_2 f) = a_{2n}(f) + 2^{23}a_{n/2}(f)$, with $a_{n/2} := 0$ for odd n.

Problem 29.3.2. $T_2 g_1 = 20468736\, g_2$ and $T_2 g_2 = g_1 + 1080\, g_2$. The characteristic polynomial of T_2 on S_{24} is $x^2 - 1080x - 20468736$, with discriminant $24^2 D$ and roots $\lambda_\pm = 540 \pm 12\sqrt{D}$. The normalized eigenforms are $f_\pm = g_1 + \lambda_\pm g_2$.

Let $\mathbf{T}_{24}$ be the $\mathbb{Z}$-algebra generated by the T_n on S_{24}.

Problem 29.3.3. $\mathbf{T}_{24} = \mathbb{Z}[T_2] = \mathbb{Z} \oplus \mathbb{Z}T_2$, and $S_{24}(\mathbb{Z}) = \mathbf{T}_{24}\, g_2$ is free of rank one over $\mathbf{T}_{24}$.

Let $\Lambda := S_{24}(\mathbb{Z}) \otimes \mathcal{O}$ in $V := S_{24}(\mathbb{Z}) \otimes K = Kf_+ \oplus Kf_-$, and let $\mathbf{T}_{24} \otimes \mathcal{O}$ act on it, through its eigenvalues, as a subalgebra of $\mathcal{O} \times \mathcal{O}$.

Problem 29.3.4. The congruence ideal of f_+ is $(\lambda_+ - \lambda_-)\mathcal{O} = 24\sqrt{D}\,\mathcal{O}$. Hence $\mathrm{C}(\Lambda; Kf_+) \cong \mathcal{O}/24\sqrt{D}\,\mathcal{O}$. Verify the module isomorphism also by direct projection.

Problem 29.3.5. $a_n(f_+) \equiv a_n(f_-) \pmod{24\sqrt{D}\,\mathcal{O}}$ for all n. The primes 2 and 3 split in K, and $\mathrm{C}(\Lambda; Kf_+)$ has length 1 at $\mathfrak{p}$, length 3 at each prime above 2, and length 1 at each prime above 3.

The congruence modulo $\mathfrak{p}$, the ramified prime of the Hecke field, is the interesting one. The primes 2 and 3 are smaller than the weight, outside the range $p > k$ of Hida's theorem below (Theorem 29.4.1).

29.4. Hida's identity

Which primes occur in the congruence module of a cuspidal eigenform? Let $k \geq 12$ be even, let $f = \sum a(n)q^n \in S_k$ be a normalized eigenform, let $K \subset \mathbb{C}$ be a number field containing the coefficients of all normalized eigenforms in S_k, $\mathcal{O} := \mathcal{O}_K$, and $\Lambda := S_k(\mathbb{Z}) \otimes \mathcal{O}$. Let $Y \subset S_k(\mathbb{Z}) \otimes K$ be the span of the other normalized eigenforms, and write $\mathrm{C}(f) := \mathrm{C}(\Lambda; Kf)$ for the decomposition $S_k(\mathbb{Z}) \otimes K = Kf \oplus Y$. Let

$$\langle g, h \rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} g(z)\,\overline{h(z)}\,y^k\,\frac{dx\,dy}{y^2}$$

be the Petersson product.

Problem 29.4.1. Y is the orthogonal complement of f, and the projection to Kf is $g \mapsto \frac{\langle g,f\rangle}{\langle f,f\rangle} f$. Hence $\mathrm{C}(f) \cong J_f/\mathcal{O}$, where J_f is the $\mathcal{O}$-module generated by the numbers $\langle g, f\rangle/\langle f, f\rangle$ with $g \in S_k(\mathbb{Z})$. For $f = f_+$ of Section 29.3, $J_f = (24\sqrt{D})^{-1}\mathcal{O}$.

The projection divides by $\langle f, f\rangle$. Hida's theorem detects the primes above $p > k$ at which the projection fails to preserve the lattice. For p prime write $a(p) = \alpha_p + \beta_p$ with $\alpha_p\beta_p = p^{k-1}$, and let

$$L(\mathrm{Sym}^2 f, s) := \prod_p \bigl(1 - \alpha_p^2 p^{-s}\bigr)^{-1}\bigl(1 - p^{k-1-s}\bigr)^{-1}\bigl(1 - \beta_p^2 p^{-s}\bigr)^{-1} \qquad (\mathrm{Re}\, s > k).$$

In the unitary normalization, the *adjoint L-function* is $L(s, \mathrm{Ad}\, f) := L(\mathrm{Sym}^2 f, s + k - 1)$; its value at $s = 1$ is $L(\mathrm{Sym}^2 f, k)$.

Problem 29.4.2. Let $\alpha, \beta \in \mathbb{C}$, $w := \alpha\beta$, and $a_m := \sum_{i=0}^m \alpha^i\beta^{m-i}$. Then, as a formal power series in X,

$$\sum_{m\geq 0} a_m^2 X^m = \frac{1 + wX}{(1 - \alpha^2 X)(1 - wX)(1 - \beta^2 X)}.$$

Hence, for $\mathrm{Re}\, s > k$, $\sum_{n\geq 1} a(n)^2 n^{-s} = \zeta(s - k + 1)\, L(\mathrm{Sym}^2 f, s)/\zeta(2s - 2k + 2)$.

Rankin's method [Ra] computes the residue of this Dirichlet series at $s = k$; in terms of $L(\mathrm{Sym}^2 f, s)$, which is holomorphic there [Sh],

$$\langle f, f\rangle = \frac{(k-1)!}{2^{2k-1}\pi^{k+1}}\, L(\mathrm{Sym}^2 f, k) \tag{1}$$

[Za, p. 109, (3)–(5)]. For $f = \Delta$ both sides equal $1.0353620568\ldots \times 10^{-6}$, the value of $\|\Delta\|^2$ found from Petersson's formula in Section 10.2 of Chapter 10.

The transcendental part of $\langle f, f\rangle$ is measured by periods. Let

$$r_f(X) := \int_0^{i\infty} f(z)(X - z)^{k-2}\,dz,$$

and let r_f^+ and r_f^- be its even and odd parts in X. By Manin's theorem there are $\omega_f^\pm \in \mathbb{C}^\times$ with $r_f^\pm \in \omega_f^\pm K[X]$ [Mn]. For $f = \Delta$ one has [KZ]

$$r_\Delta^+ = \omega^+\Bigl(\tfrac{36}{691}(X^{10} - 1) + X^2 - 3X^4 + 3X^6 - X^8\Bigr),$$
$$r_\Delta^- = \omega^-\bigl(4X^9 - 25X^7 + 42X^5 - 25X^3 + 4X\bigr).$$

Problem 29.4.3. The two polynomials in parentheses, and $X^{10}-1$, satisfy $P(X)+X^{10}P(-1/X)=0$.

The polynomial $X^{10}-1$ is a coboundary in the Eichler–Shimura cohomology, and the prime in front of it is the 691 of Problem 29.2.2. For a cusp form one discards this direction. Fix a prime $\mathfrak{p}$ of $\mathcal{O}$ above a prime $p>k$, let $\mathcal{O}_{\mathfrak{p}}$ be the localization of $\mathcal{O}$ at $\mathfrak{p}$, and let $\Omega_f^{\pm}\in\mathbb{C}^{\times}$ be Hida's periods, defined by generators of the f-eigenlines in the parabolic cohomology of $\mathrm{SL}_2(\mathbb{Z})$ with coefficients in polynomials of degree at most $k-2$ over $\mathcal{O}_{\mathfrak{p}}$; the period polynomial represents the class of f there, modulo the coboundary $X^{k-2}-1$. For the numerical checks below we use instead the displayed period-polynomial coefficients: $\Omega_f^{\pm}$ such that the coefficients of X^n in $r_f^{\pm}$ with $0<n<k-2$ lie in $\Omega_f^{\pm}\mathcal{O}_{\mathfrak{p}}$ and generate it. Identifying this normalization with Hida's in general requires an integral comparison of period polynomials and cohomology classes modulo $X^{k-2}-1$. The following theorem is due to Hida [Hi1, Hi2], with a contribution of Ribet [Ri]; see [Gh, Thm. 1]. We use it as a black box.

Theorem 29.4.1 (Hida)**.** *The number $\eta_f:=\langle f,f\rangle/(\mathrm{i}\,\Omega_f^{+}\Omega_f^{-})$ lies in K. If $\mathfrak{p}$ lies above a prime $p>k$, then $\mathrm{C}(f)\otimes\mathcal{O}_{\mathfrak{p}}\neq 0$ if and only if $\eta_f\in\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$.*

By (1), η_f is, up to a $\mathfrak{p}$-adic unit, the algebraic part $L(1,\mathrm{Ad}\,f)/(\pi^{k+1}\mathrm{i}\,\Omega_f^{+}\Omega_f^{-})$ of the adjoint L-value. In outline, the proof applies Problem 29.1.7(ii) to the parabolic cohomology lattice, self-dual for $p>k$, whose f-part is spanned by isotropic $\pm$ lines; the main difficulty is to compare this cohomological congruence module with $\mathrm{C}(f)$ [Gh, §3]; in weight 2 see [DDT, §4.4]. Two checks with PARI/GP [PARI]. For $f=\Delta$, take the coefficient of X^2 in r_{Δ}^{+} as Ω^{+} and take $\Omega^{-}=\omega^{-}$. The computation gives $\eta_{\Delta}=\pm 2^{-10}$, in agreement with the absence of cuspidal congruence primes for $S_{12}=\mathbb{C}\Delta$. Replacing Ω^{+} by $(36/691)\Omega^{+}$ would give $v_{691}(\eta_{\Delta})=1$. For $f=f_{+}$ of Section 29.3, take $K=\mathbb{Q}(\sqrt{D})$ and use the coefficients of X^2 in r_f^{+} and of X in r_f^{-} for Ω^{+} and Ω^{-}, respectively. These coefficients generate the respective fractional ideals of the interior coefficients at every prime above $p>24$. The norm $N_{K/\mathbb{Q}}(\eta_f)$ is D times a rational number whose numerator and denominator have no prime factor greater than 24. Thus $v_{\mathfrak{p}}(\eta_f)=1$ at $\mathfrak{p}=\sqrt{D}\,\mathcal{O}$, as in Problem 29.3.5.

Comparison with the full lattice. For a Hecke-stable lattice Λ as in Section 29.1, let $V_1=\bigoplus_{x\in I}V_x$ for a nonempty subset I of X, and project along the sum of the remaining eigenspaces. On the full lattice the obstruction is $\mathrm{C}(\Lambda;V_1)$, by Problems 29.1.1 and 29.1.2, and the ideal $\bigcap_{x\in I}\mathfrak{c}_x$ annihilates it, as in Problem 29.1.4. For a single cuspidal eigenline, Theorem 29.4.1 detects its congruence primes above $p>k$ by an adjoint L-value. For the Eisenstein congruence of Δ at 691, Ribet's construction gives an unramified quotient of order 691 of $A^{(679)}$ (Chapter 30). For odd primes ℓ, Chapter 27 gives the Selmer length over $\mathbb{Q}$ for positive odd twists e and identifies the classes killed by ℓ with homomorphisms on a class-group eigenspace when $(\ell-1)\nmid e$. Chapter 30 constructs a class from an Eisenstein congruence, and Problem 31.3.3 of Chapter 31 bounds the congruence length of one eigenform by a Selmer length.

The step this chapter serves. The chapter serves one step of the proof: clearing the denominators of an orthogonal projection onto a space of theta lifts. The corresponding statement of the Lean proof [GL] is `ClearsProjection`, in `grh/src/New/Determinant/Twist/FitTwists.lean`; the corresponding bound is `denominatorLe_clearingBound`, in `grh/src/New/Congruences/FirstOrder/DenominatorBound.lean`, with its specialization `denominatorLe_range_goodLift`, in `grh/src/New/Congruences/FitTwistsExist.lean`.

Exercise. For $k = 16, 18, 20, 22, 26$ the space S_k is spanned by one normalized eigenform f_k, and $a_\ell(f_k) \equiv 1 + \ell^{k-1} \pmod{N_k}$ for every prime ℓ.

Hints

§29.1.

29.1.1: Consider $\Lambda \to \pi_1(\Lambda)/(\Lambda \cap V_1)$, $\lambda \mapsto \pi_1(\lambda)$, and compute its kernel.
29.1.2: The map $\pi_1(\Lambda) \to (\Lambda + \pi_1(\Lambda))/\Lambda$.
29.1.3: $\pi_1(\Lambda) = \mathbb{Z}e_1$.
29.1.4: Apply to Λ an element of $\mathbf{T}$ that vanishes off x. For the length, $\pi_x(\Lambda)$ is free of rank $\dim V_x$.
29.1.5: Identify V_x with K by $t \mapsto t(x)$; then $\pi_x(\mathbf{T}) = \mathcal{O}$.
29.1.6: $\Lambda \cap V_1$ is a direct summand of Λ, so an $\mathcal{O}$-valued functional on it extends to Λ; then use that the pairing is perfect.
29.1.7: Compute the dual basis.

§29.2.

29.2.1: Subtract a multiple of E_4^3 to kill the constant term; $S_{12} = \mathbb{C}\Delta$.
29.2.2: Compare the coefficients of 1 and q. Then use $\gcd(432000, 691) = 1$.
29.2.3: The projection to $\mathbb{Q}E_k$ is $g \mapsto a_0(g)E_k$, and some $E_4^a E_6^b$ lies in $M_k(\mathbb{Z})$ with constant term 1.
29.2.4: For the first claim use the formula for $a_n(T_m f)$ in [SeA, Ch. VII, §5.3]. For the second, Problem 29.2.2 gives one inclusion; for the other use T_2, T_3 and $\gcd(2073, 176896) = 691$. Here $\tau(2) = -24$ and $\tau(3) = 252$.
29.2.5: For $g = aE_4^3 + b\Delta$, compute the index of $\mathbf{T}_{12}\, g$ in $M_{12}(\mathbb{Z})$ using $(1, 1)$ and $(691, 0)$; note $432000 \equiv 125 \pmod{691}$.

§29.3.

29.3.1: As in Problem 29.2.1, with $S_{24} = \Delta M_{12}$.
29.3.2: By Problem 29.3.1 the first two coefficients of $Ag_1 + Bg_2$ are A, B; a normalized eigenform has $a_1 = 1$ and $a_2 =$ its T_2-eigenvalue [SeA, Ch. VII, §5.4]. One has $a_4(g_1) = 12080128$ and $a_4(g_2) = 1080$.
29.3.3: $\mathbf{T}_{24} \otimes \mathbb{Q} = \mathbb{Q}[T_2]$, and the matrix of T_2 in the basis g_1, g_2 has an entry 1.
29.3.4: Write $t = a + bT_2$ with $a, b \in \mathcal{O}$, and apply Problem 29.1.5. For the direct check, write g_1, g_2 in terms of $f_\pm$.
29.3.5: Write $f_+ - f_-$ in the basis g_1, g_2. For the splitting, reduce D modulo 8 and 3.

§29.4.

29.4.1: Use simultaneous Hecke eigenspaces. For the integral lattice use a basis with $a_i(g_j) = \delta_{ij}$ in the first $\dim S_k$ coefficients, and recall $a_1(f) = 1$. For the example use Problem 29.3.4.
29.4.2: First take $\alpha \neq \beta$ and use the difference of powers: expand a_m^2 as a combination of α^{2m}, w^m and β^{2m}, and sum three geometric series. Both sides of the resulting coefficient identities are polynomials in α, β.

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