BLACK BOX: GALOIS REPRESENTATIONS OF EIGENFORMS

Prerequisites. Local and global class field theory, including ray class groups and the Chebotarev density theorem [Ne, Ch. V–VI, §VII.13]. Elliptic curves at the level of [Si1, Ch. III, V, §§VII.3–VII.4], and the statement of Tate's uniformization [Si2, Ch. V]; the curve $y^2 = x^3 - x$ as in Chapter 13, *Complex multiplication and reduction at an inert prime*. Modular forms of weight 2 and Hecke operators [DS, Ch. 1–6]. Lattices over a discrete valuation ring. The exercises use the cyclotomic character to illustrate Hodge–Tate weights; crystallinity and potential semistability enter only through the stated theorems. The cited theorems are black boxes for this chapter. Section 32.4 states a theorem of the proof and gives the definitions needed to read it.

32.1. Galois representations over CM fields

What Galois representation does an automorphic representation determine, and at which places can the two be compared?

In this chapter F is any CM field (not the field E of Chapter 33), with maximal totally real subfield F^+ and complex conjugation $x \mapsto \bar{x}$; ℓ is a prime and $\iota\colon \mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ a ring isomorphism ([BLGGT] use its inverse). Let ϵ_ℓ be the ℓ-adic cyclotomic character. A cuspidal automorphic representation π of $\mathrm{GL}_a(\mathbb{A}_F)$ is *polarizable* if $\pi^c \cong \pi^\vee \otimes (\chi \circ N_{F/F^+} \circ \det)$ for a continuous character $\chi\colon \mathbb{A}_{F^+}^\times/(F^+)^\times \to \mathbb{C}^\times$ with $\chi_v(-1) = (-1)^a$ at every real place v, and *regular algebraic* if π_∞ has the infinitesimal character of an irreducible algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_a$; see [BLGGT, §2.1]; the pair (π, χ) is then called polarized. For a finite place w with residue field of order q_w and uniformizer ϖ_w, an irreducible representation π_w of $\mathrm{GL}_a(F_w)$ is *unramified* if it has a nonzero vector fixed by $\mathrm{GL}_a(\mathcal{O}_{F_w})$; it is then a subquotient of the normalized induction of the character $b \mapsto \prod_{i=1}^a \chi_i(b_{ii})$ of the upper triangular Borel subgroup, where the χ_i are unramified characters of $F_w^\times$, and its *Satake parameters* are $\chi_1(\varpi_w), \dots, \chi_a(\varpi_w)$.

Theorem 32.1.1. *Let (π, χ) be a regular algebraic polarized cuspidal pair as above. There is a continuous semisimple $r = r_{\ell,\iota}(\pi)\colon G_F \to \mathrm{GL}_a(\overline{\mathbb{Q}}_\ell)$ such that:* (i) $r^c \cong r^\vee \otimes \epsilon_\ell^{1-a} r_{\ell,\iota}(\chi)|_{G_F}$*;* (ii) *at every finite place $w \nmid \ell$ at which π_w is unramified, r is unramified, and for a geometric Frobenius* Frob_w *the eigenvalues of $r(\mathrm{Frob}_w)$ are $\iota(q_w^{(a-1)/2}\alpha_1), \dots, \iota(q_w^{(a-1)/2}\alpha_a)$, where $\alpha_1, \dots, \alpha_a$ are the Satake parameters of π_w;* (iii) *for every $w \mid \ell$, $r|_{G_{F_w}}$ is potentially semistable, crystalline if π_w is unramified, with Hodge–Tate weights determined by π_∞.*

The construction uses the cohomology of unitary Shimura varieties and, in the generality treated by Chenevier and Harris, also p-adic families [Cl, HT, Shi, CH]. At the ramified places $w \nmid \ell$ the published theorem gives more: the Frobenius-semisimplified Weil–Deligne representation of $r|_{G_{F_w}}$ corresponds, after applying ι^{-1}, to $\pi_w \otimes |\det|_w^{(1-a)/2}$ under the local Langlands correspondence; in this generality this is [BLGGT, Thm. 2.1.1], incorporating results of Taylor–Yoshida and

Date: November 26, 2026.

Caraiani [TY, Car1]. For part (iii) see also [BLGGT, Thm. 2.1.1] and the work cited there, including [Car2]. Section 32.4 states the theorem on Galois representations of eigenforms that the proof establishes for definite unitary groups.

The smallest case is $a = 1$ over an imaginary quadratic field K, where the unitary group is $U(1) := \ker(N\colon \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m)$.

Problem 32.1.1. $U(1)(\mathbb{R})$ is the unit circle. $U(1)(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$ if p splits in K, and $U(1)(\mathbb{Q}_p)$ is compact otherwise.

Problem 32.1.2. Let ϕ be a character of $\mathbb{A}_K^\times/K^\times$ that is trivial on $\mathbb{A}_\mathbb{Q}^\times$. Then $\phi^c = \phi^{-1}$, and there is exactly one character μ of $U(1)(\mathbb{A})/U(1)(\mathbb{Q})$ such that $\phi(x) = \mu(x/x^c)$ for every $x \in \mathbb{A}_K^\times$. Conversely each such μ gives such a ϕ.

Problem 32.1.3. Let ϕ be a character of $\mathbb{A}_K^\times/K^\times$. If $\phi^c = \phi^{-1}$, then $\phi|_{\mathbb{A}_\mathbb{Q}^\times}$ is either trivial or the quadratic character $\omega_{K/\mathbb{Q}}$.

So the characters with $\phi^c = \phi^{-1}$ fall into two classes, and by Problem 32.1.2 only the first comes from $U(1)$.

32.2. The Hecke character of $\mathbb{Q}(\mathrm{i})$

What is the representation of Theorem 32.1.1 when $a = 1$, and how does it compare with the curve $E\colon y^2 = x^3 - x$? Let $K = \mathbb{Q}(\mathrm{i})$, $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$ and $\mathfrak{f} = (2 + 2\mathrm{i})$.

Problem 32.2.1. Every ideal $\mathfrak{a}$ of $\mathcal{O}$ prime to $1 + \mathrm{i}$ has exactly one generator $\alpha \equiv 1 \pmod{\mathfrak{f}}$. Let $\psi(\mathfrak{a}) := \alpha$. Then ψ is multiplicative, $\psi(\mathfrak{a}^c) = \overline{\psi(\mathfrak{a})}$, and $\psi(\mathfrak{a})\psi(\mathfrak{a}^c) = N\mathfrak{a}$.

For $p \equiv 1 \pmod 4$ and $\mathfrak{p} \mid p$, one has $p+1-\#E(\mathbb{F}_p) = \psi(\mathfrak{p})+\psi(\mathfrak{p}^c)$ [IR, Ch. 18, §4].

Problem 32.2.2. $\psi((1 + 2\mathrm{i})) = -1 - 2\mathrm{i}$ and $\psi((3 + 2\mathrm{i})) = 3 + 2\mathrm{i}$, and $\#E(\mathbb{F}_5) = \#E(\mathbb{F}_{13}) = 8$.

Problem 32.2.3. Let $n \geq 1$. If ideals $\mathfrak{a}, \mathfrak{b}$ prime to 2ℓ have the same class in the ray class group modulo $\mathfrak{f}\ell^n$, then $\psi(\mathfrak{a}) \equiv \psi(\mathfrak{b}) \pmod{\ell^n\mathcal{O}}$.

Combining Problem 32.2.3 with Artin reciprocity for the ray class fields modulo $\mathfrak{f}\ell^n$ gives the following.

Theorem 32.2.1. *There is a continuous character $\psi_\ell\colon G_K \to (\mathcal{O} \otimes \mathbb{Z}_\ell)^\times$, unramified outside 2ℓ, with $\psi_\ell(\mathrm{Frob}_\mathfrak{p}) = \psi(\mathfrak{p})$ for every prime $\mathfrak{p} \nmid 2\ell$, where $\mathrm{Frob}_\mathfrak{p}$ is an arithmetic Frobenius.*

By Problem 32.2.1, $\psi\psi^c$ is the norm. For a $\mathbb{Z}_\ell$-algebra map $\tau\colon \mathcal{O} \otimes \mathbb{Z}_\ell \to \overline{\mathbb{Q}}_\ell$, the character $\tau \circ \psi_\ell$ takes a geometric Frobenius at $\mathfrak{p} \nmid 2\ell$ to $\tau(\psi(\mathfrak{p}))^{-1}$; in Theorem 32.1.1(ii) the corresponding unramified local character takes $\varpi_\mathfrak{p}$ to $\iota^{-1}(\tau(\psi(\mathfrak{p}))^{-1})$. At the places not above ℓ the theorem is then class field theory; the next problem looks at the places above ℓ.

Problem 32.2.4. Let $\ell \equiv 1 \pmod 4$, $\ell = \lambda\bar{\lambda}$, and let $\psi_\lambda\colon G_K \to \mathbb{Z}_\ell^\times$ be ψ_ℓ followed by $\mathcal{O} \otimes \mathbb{Z}_\ell \to \mathcal{O}_\lambda = \mathbb{Z}_\ell$. Then $\psi_\lambda = \epsilon_\ell$ on a subgroup containing a neighborhood of the identity in the inertia group at λ, and ψ_λ has finite image on the inertia group at $\bar{\lambda}$.

So ψ_λ has Hodge–Tate weights 1 and 0 at the two places above ℓ (in the convention where ϵ_ℓ has weight 1), and the polarization appears as $\mathrm{HT}_{\tau c} = 1 - \mathrm{HT}_\tau$.

The automorphism $[\mathrm{i}](x, y) = (-x, \mathrm{i}y)$ of E is defined over K, so $V_\ell(E) := T_\ell(E) \otimes \mathbb{Q}$ is free of rank one over $K \otimes \mathbb{Q}_\ell$, and G_K acts on it through a character $G_K \to (K \otimes \mathbb{Q}_\ell)^\times$.

Since E has good reduction at $\mathfrak{p} \nmid 2$, $V_\ell(E)$ is unramified at $\mathfrak{p} \nmid 2\ell$ [Si1, §VII.4], and an element $\beta \in \mathcal{O}$ acts on $V_\ell(E)$ through $[\beta]$.

Problem 32.2.5. Let $\mathfrak{p} \nmid 2\ell$, and let $\mathrm{Frob}_\mathfrak{p}$ denote an arithmetic Frobenius. If $N\mathfrak{p} = p \equiv 1 \pmod 4$, then $\mathrm{Frob}_\mathfrak{p}$ acts on $V_\ell(E)$ as $\psi(\mathfrak{p})$. If $\mathfrak{p} = (p)$ with $p \equiv 3 \pmod 4$, it acts as $-p = \psi((p))$.

By the Chebotarev density theorem, Problem 32.2.5 proves the following; it is due to Deuring, see [Si2, Ch. II, §10].

Theorem 32.2.2. *The character of G_K on $V_\ell(E)$ is ψ_ℓ.*

Problem 32.2.6. Let $\psi_1, \psi_2 \colon G_K \to \overline{\mathbb{Q}}_\ell^\times$ be ψ_ℓ followed by the two $\mathbb{Z}_\ell$-algebra homomorphisms $\mathcal{O} \otimes \mathbb{Z}_\ell \to \overline{\mathbb{Q}}_\ell$. Then $\psi_2 = \psi_1^c$, $V_\ell(E) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \psi_1$, and $\mathrm{tr}(\mathrm{Frob}_p \mid V_\ell(E)) = 0$ for $p \equiv 3 \pmod 4$, $p \neq \ell$.

Over $\overline{\mathbb{Q}}_\ell$, the restriction of $V_\ell(E) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ to G_K is $\psi_1 \oplus \psi_1^c$, which is reducible. So the weight-two form of E, base changed to K, is not cuspidal, and it enters Theorem 32.1.1 only through ψ, with $a = 1$. For a cuspidal example with $a = 2$ one needs a curve without complex multiplication by K. We take $X_0(11)$ and stay over $\mathbb{Q}$; its form has no complex multiplication, hence no quadratic self-twist, so its base change to K is cuspidal [AC, Ch. 3], and the local questions are the same.

32.3. Eichler–Shimura for $X_0(11)$

How does one get Frobenius eigenvalues out of the cohomology of a Shimura variety, and what does each method give? Let

$$f = q \prod_{n \geq 1} (1 - q^n)^2 (1 - q^{11n})^2 = \sum_{n \geq 1} a_n q^n,$$

the normalized eigenform spanning $S_2(\Gamma_0(11))$ [DS]. The curve $X_0(11)$ has genus one and the model $E \colon y^2 + y = x^3 - x^2 - 10x - 20$ over $\mathbb{Q}$, with good reduction away from 11 (LMFDB, curve 11.a2, `https://www.lmfdb.org/EllipticCurve/Q/11/a/2`). For $p \neq 11$, the Hecke operator T_p acts on $E = J_0(11)$ as $[a_p]$, and by the Eichler–Shimura relation [DS, Thm. 8.7.2], [Sh, Thm. 7.9], the reduction of T_p modulo p is $F + V$, where F is the Frobenius endomorphism and $FV = VF = p$.

Problem 32.3.1. $a_2 = -2$, $a_3 = -1$, $a_5 = 1$, $a_7 = -2$, $a_{11} = 1$, $a_{13} = 4$.

Problem 32.3.2. For $p \notin \{11, \ell\}$, the arithmetic Frobenius at p on $V_\ell(E)$ satisfies $\mathrm{Frob}_p^2 - a_p \mathrm{Frob}_p + p = 0$.

For $p \notin \{11, \ell\}$, the arithmetic Frobenius at p has determinant p on $V_\ell(E)$, by the Weil pairing [Si1, §III.8], and $\#E(\mathbb{F}_{p^k}) = p^k + 1 - \mathrm{tr}(\mathrm{Frob}_p^k \mid V_\ell(E))$ for $k \geq 1$ [Si1, §V.2].

Problem 32.3.3. Let $\Phi \in \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ satisfy $\Phi^2 - a\Phi + p = 0$ with $a \in \mathbb{Z}$, and let $\det \Phi = p$. Then the characteristic polynomial of Φ is $X^2 - aX + p$. Hence, taking $\ell \neq p$, $\#E(\mathbb{F}_p) = p + 1 - a_p$ for $p \neq 11$.

The relation of Problem 32.3.2 kills Frobenius; it does not give the multiplicities of its roots. In dimension two the determinant supplies them. In dimension three, $\mathrm{diag}(\alpha,\alpha,\beta)$ and $\mathrm{diag}(\alpha,\beta,\beta)$ are killed by the same quadratic polynomial. In the cohomological construction, counts of points give traces of powers of Frobenius, following Langlands and Kottwitz [Ko]. The simplest instance is the trace formula for $\#E(\mathbb{F}_{p^k})$ stated before Problem 32.3.3.

Problem 32.3.4. $\#E(\mathbb{F}_p) = p + 1 - a_p$ for $p = 2, 3, 5, 7, 13$ by direct count. For $p \notin \{11, \ell\}$, the numbers $\#E(\mathbb{F}_p)$ and $\#E(\mathbb{F}_{p^2})$ determine the characteristic polynomial of the arithmetic Frobenius on $V_\ell(E)$; for $p = 2$ both equal 5.

At 11 the curve has split multiplicative reduction. By Tate's uniformization [Si2, Ch. V], E is uniformized over $\mathbb{Q}_{11}$ by $\mathbb{G}_m/q^{\mathbb{Z}}$ for some nonzero $q \in 11\mathbb{Z}_{11}$, and the exact sequence $0 \to \mathbb{Z}_\ell(1) \to T_\ell(E) \to \mathbb{Z}_\ell \to 0$ has extension class given by the Kummer class of q.

Problem 32.3.5. For $\ell \neq 11$, the inertia group at 11 acts on $V_\ell(E)$ through a nontrivial group of unipotent matrices; in particular $V_\ell(E)$ is ramified at 11.

So the place 11 is not of the kind covered by Theorem 32.1.1(ii); the finer compatibility there, through the Weil–Deligne representation, is due to Carayol [Ca]. The next section asks what survives if one compares only at unramified places.

32.4. Hecke polynomials on a definite unitary group

The input to Theorem 32.1.1 is a representation of $\mathrm{GL}_a(\mathbb{A}_F)$. Theorem 32.4.1 starts with forms on the unitary group of the standard definite hermitian space. What does an eigenform there determine at a split place, and in what sense is it compared with Frobenius?

Let F be a CM field, $n \geq 1$, and

$$G_f := \{g \in \mathrm{GL}_n(\mathbb{A}_{F,f}) : \bar{g}^T g = 1\}, \qquad \mathcal{K}_0 := G_f \cap \mathrm{GL}_n(\widehat{\mathcal{O}}_F).$$

For a finite place w of F, let q_w be the order of its residue field, ϖ_w a uniformizer of F_w, and $\mathcal{K}_w$ the group of $k \in \mathcal{K}_0$ whose components at all places other than $w, \bar{w}$ are 1. Let w be *split*, $\bar{w} \neq w$. For $a \in \mathrm{GL}_n(F_w)$ let $b_w(a) \in G_f$ have component a at w, $(\bar{a}^T)^{-1}$ at $\bar{w}$ and 1 elsewhere. For $j \geq 0$ let

$$D_j := \{a \in \mathrm{GL}_n(F_w) : a \text{ and } \varpi_w a^{-1} \text{ have entries in } \mathcal{O}_w,\ \mathrm{ord}_w \det a = j\}.$$

For a function ϕ on G_f fixed by $\mathcal{K}_w$ under right translation, the Hecke operator is $(T_{w,j}\phi)(g) := \sum_a \phi(g\, b_w(a))$, the sum over a in a set of representatives of $D_j/\mathrm{GL}_n(\mathcal{O}_w)$. For $q, N \in \mathbb{N}$ and $t\colon \mathbb{N} \to \mathbb{C}$ put

$$H(q, N, t) := \sum_{j=0}^{N} (-1)^j q^{\binom{j}{2}}\, t(j)\, X^{N-j}.$$

A function ϕ on G_f with values in a finite-dimensional complex vector space *has Hecke polynomial* $Q \in \mathbb{C}[X]$ at the split place w if ϕ is fixed by $\mathcal{K}_w$ and there is $t\colon \mathbb{N} \to \mathbb{C}$ with $Q = H(q_w, n, t)$ and $T_{w,j}\phi = t(j)\phi$ for every $j \in \mathbb{N}$.

Problem 32.4.1. The map $a \mapsto a\mathcal{O}_w^n/\varpi_w\mathcal{O}_w^n$ is a bijection from $D_j/\mathrm{GL}_n(\mathcal{O}_w)$ onto the subspaces of codimension j of $\mathbb{F}_{q_w}^n$. Hence $D_j = \emptyset$ for $j > n$, $D_n = \varpi_w \mathrm{GL}_n(\mathcal{O}_w)$, and D_1 is the union of $q_w + 1$ cosets when $n = 2$.

Problem 32.4.2. Let $n = 1$. Then $T_{w,0}\phi = \phi$, $(T_{w,1}\phi)(g) = \phi(g\, b_w(\varpi_w))$ and $T_{w,j}\phi = 0$ for $j \geq 2$. A nonzero ϕ fixed by $\mathcal{K}_w$ has a Hecke polynomial at w if and only if $\phi(g\, b_w(\varpi_w)) = t_1\phi(g)$ for a constant t_1 and all g, and then the polynomial is $X - t_1$.

Problem 32.4.3. $H(q, 2, t) = t(0)X^2 - t(1)X + q\, t(2)$. If $q \geq 1$, $\beta_1, \ldots, \beta_N \in \mathbb{C}$, and $t(j) = q^{-\binom{j}{2}} e_j(\beta_1, \ldots, \beta_N)$ for $0 \leq j \leq N$, with e_j the elementary symmetric polynomials, then $H(q, N, t) = \prod_{i=1}^{N}(X - \beta_i)$.

Away from ℓ, the split-place clause of Theorem 32.4.1 below evaluates the local polynomial on a *geometric* Frobenius, whose inverse acts on the residue field by $x \mapsto x^{q_w}$. For $E = X_0(11)$ and $p \notin \{11, \ell\}$, taking $t = (1, a_p, 1, 0, 0, \ldots)$ gives $H(p, 2, t) = X^2 - a_p X + p$, which kills the arithmetic Frobenius on $V_\ell(E)$ by Problem 32.3.2.

Problem 32.4.4. Let $\Phi \in \mathrm{GL}(V)$ satisfy $\Phi^2 - a\Phi + p = 0$, and let $\mathrm{GL}(V)$ act on the dual $V^\vee$ by $g \mapsto (g^{-1})^T$. Then Φ^{-1} acts on $V^\vee$ by a matrix that satisfies the same equation. So for $p \notin \{11, \ell\}$, $X^2 - a_p X + p$ kills the geometric Frobenius at p on $V_\ell(E)^\vee$.

In rank one the forms of trivial weight are the complex-valued functions on $G_f = \{x \in \mathbb{A}_{F,f}^\times : x\bar{x} = 1\}$ that are invariant under $U(1)(F^+) := \{\gamma \in F^\times : \gamma\bar{\gamma} = 1\}$, embedded diagonally, and whose stabilizer under right translation contains a neighborhood of the identity.

Problem 32.4.5. Let $n = 1$ and let μ be a character of G_f, trivial on $U(1)(F^+)$, whose kernel contains a neighborhood of the identity. Then μ is a form of trivial weight, and at every split w with $\mu(b_w(\mathcal{O}_w^\times)) = 1$ it has Hecke polynomial $X - \mu(b_w(\varpi_w))$. Here $b_w(x)$ has components x at w and $\bar{x}^{-1}$ at $\bar{w}$.

For an eigenform, all but finitely many places of prime norm must be split and admit the prescribed Hecke polynomial.

Problem 32.4.6. In $\mathbb{Q}(\mathrm{i})$ the place $(1+\mathrm{i})$ has norm 2 and is fixed by conjugation. If a CM field F contains a root of unity of exact order 5040, then every finite place v of F with q_v prime is split.

A homomorphism r from a group Γ to $\mathrm{GL}_n(k)$, k a field, is *trace-nondegenerate* if every z in the k-span of $r(\Gamma)$ with $\mathrm{tr}(z\, r(h)) = 0$ for all $h \in \Gamma$ is 0.

Problem 32.4.7. Every character $\Gamma \to k^\times$ is trace-nondegenerate. The homomorphism $r\colon \mathbb{Z} \to \mathrm{GL}_2(k)$, $m \mapsto \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$, is not trace-nondegenerate.

The statement used in the proof. Fix ℓ and ι as in Section 32.1, an algebraic closure $\overline{\mathbb{Q}}$, and a CM field F with an embedding $F \to \overline{\mathbb{Q}}$ and a CM type $\Phi = (\Phi_v)_{v|\infty}$, an embedding $\Phi_v\colon F \to \mathbb{C}$ inducing each infinite place v. Let $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Assume F contains a root of unity of exact order 5040. *Weil data* on such a field consist of an element $\jmath \in F$ with $\bar{\jmath} = -\jmath$, $\mathrm{Im}\,\Phi_v(\jmath) > 0$ for every $v \mid \infty$, and $(1 - \jmath^{-1})/2$ integral; a finite idele α with $\mathrm{ord}_w(\jmath\alpha\bar{\alpha}) = -\mathrm{ord}_w(\mathfrak{D}_{F/\mathbb{Q}})$ at every finite w, $\mathfrak{D}_{F/\mathbb{Q}}$ the different; and a continuous character υ of $\mathbb{A}_F^\times/F^\times$ with values of absolute value 1, such that $\upsilon(x) = \prod_{v|\infty} x_v/|x_v|$ whenever the finite part of x lies in $\widehat{\mathcal{O}}_F^\times$ (with $x_v \in \mathbb{C}^\times$ through Φ_v), $\upsilon(\bar{x}) = \upsilon(x)^{-1}$ for all x, and $\upsilon(\varpi_w) = -1$ at every w with $\bar{w} = w$ (ϖ_w placed at w, 1 elsewhere).

In this chapter a *weight* $\kappa = (c, e)$ is the weight of a form (not the weight of a place of Chapter 33): functions $c_v \colon \mathbb{N} \to \mathbb{N}$ and integers e_v, for $v \mid \infty$. Let W_κ be the tensor product over v of the representations of $\mathrm{GL}_n(\mathbb{C})$ with $c_v(h)$ columns of height h, $0 \le h < n$, twisted by $\det^{e_v}$, written in coordinates through Schur matrices in chosen bases of tableaux, and let $U(F^+) := \{\gamma \in \mathrm{GL}_n(F) : \bar{\gamma}^T \gamma = 1\}$ act on it through the complex conjugates of the $\Phi_v(\gamma)$. A *form of weight* κ is a function $f \colon G_f \to W_\kappa$ whose stabilizer under right translation contains a neighborhood of the identity and which satisfies $f(\gamma g) = \gamma \cdot f(g)$ for $\gamma \in U(F^+)$, embedded diagonally, and $g \in G_f$. For a family $P = (P_v)$ of polynomials in $\mathbb{C}[X]$ indexed by the finite places of F, f is an *eigenform for* P if $f \neq 0$ and, for all but finitely many finite places v, if q_v is prime then v is split and f has Hecke polynomial P_v at v.

A *Frobenius datum* (v, A, s) is a finite place v of F with q_v prime, a valuation ring A of $\overline{\mathbb{Q}}$ lying over v (an $x \in \mathcal{O}_F$ is in the maximal ideal of A exactly when $x \in v$), and $s \in G_F$ such that s^{-1} stabilizes A and induces $x \mapsto x^{q_v}$ on its residue field. A set Y of Frobenius data contains *most Frobenius data* if each nonempty member $\mathcal{U}$ of the topology of G_F contains a nonempty member $\mathcal{U}'$ of that topology such that, for all but finitely many v, Y contains every datum (v, A, s) with $s \in \mathcal{U}'$. A homomorphism $r \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is *attached to* P if it is continuous and trace-nondegenerate, and for every real $\eta > 0$ the set of Frobenius data (v, A, s) for which every coefficient of $\det(X - r(s)) - \iota P_v$ has ℓ-adic absolute value at most η contains most Frobenius data; here ιP_v is P_v with ι applied to its coefficients.

Theorem 32.4.1. *Let F have Weil data, let $n \ge 1$, and let w be a finite place of F with $w \nmid \ell$. For every family P, every weight κ and every form f of weight κ that is an eigenform for P:*

(a) *some $r \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ is attached to P;*
(b) *for every r attached to P, if f is fixed by $\mathcal{K}_w$, then for every valuation ring A of $\overline{\mathbb{Q}}$ over w:*
  (i) *r is trivial on the inertia group of A;*
  (ii) *if w is split, then for every $Q \in \mathbb{C}[X]$ such that f has Hecke polynomial Q at w, and every $s \in G_F$ such that s^{-1} stabilizes A and induces $x \mapsto x^{q_w}$ on its residue field, $(\iota Q)(r(s)) = 0$.*

The full theorem gives (a) for every finite place w. At $w \mid \ell$, if f is fixed by $\mathcal{K}_w$, it gives crystallinity and the Hodge–Tate jumps determined by κ for each attached r at each A over w. At a split such w, a local Hecke polynomial also gives a condition on a crystalline Frobenius. We omit the precise statements at $w \mid \ell$. In (b)(ii), ιQ kills $r(s)$. This alone does not determine the multiplicities in the characteristic polynomial (compare $\mathrm{diag}(\alpha, \alpha, \beta)$ in Section 32.3).

In the Lean formalization. Theorem 32.4.1 is the case $w \nmid \ell$ of `GaloisRep.sphericalCompat_std` of the Lean proof [GL] (names omit the namespace `New`), with conclusion `GaloisRep.SphericalCompat`; its local conclusion at each A over w is `GaloisRep.IsCompatibleAt`, which at $w \nmid \ell$ reduces to (i) and (ii). The Lean proof obtains the theorem by lifting f along two characters to rank $2n + 1$, attaching a representation to each lift, twisting each, and cancelling the $n + 1$ extra characters on each side.

Problem 32.4.8. Let $n = 1$, $\kappa = (0, 0)$ (that is, $c_v = 0$ and $e_v = 0$ for all v, so $W_\kappa = \mathbb{C}$), and let F contain a root of unity of exact order 5040. The constant function $f = 1$ is a form of weight κ and an eigenform for $P_v = X - 1$ (all v), and

the trivial homomorphism $r = 1$ is attached to P. At every split $w \nmid \ell$ and for every s as in Theorem 32.4.1(b)(ii), $(\iota Q)(r(s)) = 0$ for $Q = X - 1$.

Problem 32.4.9. Let F have Weil data, let f be a form that is an eigenform for P, and let r be attached to P. Then the set of Frobenius data (v, A, s) with $(\iota P_v)(r(s)) = 0$ contains most Frobenius data.

Thus $(\iota P_v)(r(s)) = 0$ for most Frobenius data (v, A, s): the Hecke polynomial kills Frobenius. This is the step that the last part, from Chapter 33 on, takes from this chapter.

Hints

§32.1.

32.1.1: $K \otimes \mathbb{Q}_p$ is $\mathbb{Q}_p \times \mathbb{Q}_p$ or a field.

32.1.2: Hilbert's Theorem 90 for K and for each $K \otimes \mathbb{Q}_v$, and for the units at the unramified places.

32.1.3: ϕ is trivial on $\mathbb{Q}^\times$ and on the norms xx^c; use $\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times N(\mathbb{A}_K^\times) \cong \mathbb{Z}/2$.

§32.2.

32.2.1: The units $\pm 1, \pm \mathrm{i}$ map bijectively onto $(\mathcal{O}/\mathfrak{f})^\times$, which has order 4; and $\mathfrak{f}^c = \mathfrak{f}$.

32.2.3: Write $\mathfrak{a} = \mathfrak{b}(\beta)$ with $\beta \equiv 1 \pmod{{}^\times \mathfrak{f}\ell^n}$; compare $\beta\,\psi(\mathfrak{b})$ with the primary generator of $\mathfrak{a}$.

32.2.4: Approximate $u \in 1 + \ell^m\mathbb{Z}_\ell$ at λ by Gaussian integers $\equiv 1$ at $\bar{\lambda}$ and modulo a high power of $1 + \mathrm{i}$; local reciprocity, normalized as in Theorem 32.2.1.

32.2.5: In the split case use the action on the invariant differential to distinguish the two primes over p, and the trace to distinguish unit multiples. In the inert case compare Frobenius over $\mathbb{F}_{p^2}$ with Frobenius over $\mathbb{F}_p$.

32.2.6: Complex conjugation acts on $V_\ell(E)$ and anticommutes with [i].

§32.3.

32.3.1: Expand the product to order q^{13}.

32.3.2: Multiply $T_p = F + V$ by F, and use that Frob_p acts on T_ℓ of the reduction as F.

32.3.3: Cayley–Hamilton.

32.3.4: In $\mathbb{F}_4$, the equation becomes $y^2 + y = x^3 + x^2$. Use $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$.

32.3.5: Kummer theory for the extensions $\mathbb{Q}_{11}^{\mathrm{ur}}(q^{1/\ell^n})$; $v_{11}(q) \neq 0$.

§32.4.

32.4.1: The lattice $a\mathcal{O}_w^n$ determines the coset $a\mathrm{GL}_n(\mathcal{O}_w)$; it lies between $\varpi_w\mathcal{O}_w^n$ and $\mathcal{O}_w^n$, with index q_w^j in $\mathcal{O}_w^n$.

32.4.2: Problem 32.4.1 with $n = 1$.

32.4.3: Expand the product and compare coefficients.

32.4.4: Transpose the equation for Φ.

32.4.5: $\mathcal{K}_w = b_w(\mathcal{O}_w^\times)$, and μ is multiplicative.

32.4.6: If $\bar{v} = v$, conjugation induces an automorphism of the residue field $\mathbb{F}_p$; take a root ζ of order 5040 and consider $\zeta^2 - 1$; $1 - \zeta^2$ is a unit because 2520 is not a prime power.

32.4.7: For a character the span is k. For the second assertion, consider $r(1) - r(0)$.

32.4.8: Problems 32.4.2, 32.4.6 and 32.4.7.

32.4.9: Theorem 32.4.1(b)(ii) at $w = v$, $Q = P_v$, away from finitely many places.

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