POINCARÉ SERIES AND NEAR ORTHOGONALITY

Prerequisites. Modular forms for $\mathrm{SL}_2(\mathbb{Z})$: q-expansions, the cusp form Δ and the dimension formula [SeA, Ch. VII, §3]. The residue theorem and the mean value property of holomorphic functions. The power series $J_\nu(x) = \sum_{j\ge0}(-1)^j(x/2)^{\nu+2j}/(j!\,\Gamma(\nu+j+1))$. Gershgorin's theorem on the eigenvalues of a matrix. Chapter 9, *Few translates span*, for the closing paragraphs only.

Chapter 9 asked for a map into a space of automorphic forms that is almost an isometry. If the weight is large, Poincaré series of small index are almost orthogonal, and Petersson's formula proves it with nothing more than a bound for a Bessel function. This chapter works it out on $\mathrm{SL}_2(\mathbb{Z})$, where the vectors are holomorphic Poincaré series and their Petersson inner products recover Fourier coefficients up to an explicit factor.

Throughout, $k \ge 4$ is even, $\Gamma := \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_\infty := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$, $j(\gamma, z) := cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $e(z) := e^{2\pi i z}$, and $d\nu := y^{-2}\,dx\,dy$ on the upper half-plane $\mathbb{H}$.

10.1. Poincaré series

Let S_k be the space of cusp forms of weight k for Γ, with the Petersson inner product $\langle f, g\rangle := \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}\,y^k\,d\nu(z)$. For $f \in S_k$ write $f(z) = \sum_{n\ge1} a_f(n)\,e(nz)$. For each positive integer m the *Poincaré series*

$$P_m(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k}\,e(m\gamma z)$$

converges absolutely and uniformly on compact sets, and $P_m \in S_k$ [Iw, §§3.1–3.2].

Problem 10.1.1. For $f \in S_k$ and $m \ge 1$,

$$\langle f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\,a_f(m).$$

Problem 10.1.2. The series P_m, $m \ge 1$, span S_k.

So $P_m = 0$ for $k \in \{4, 6, 8, 10, 14\}$, where $S_k = 0$, although no term of the series vanishes. The first nonzero case is $k = 12$, where $S_{12} = \mathbb{C}\Delta$ with $\Delta = \sum_{n\ge1}\tau(n)\,e(nz)$.

Problem 10.1.3. For $k = 12$,

$$P_m = \frac{10!\,\tau(m)}{(4\pi m)^{11}\,\|\Delta\|^2}\,\Delta.$$

In particular P_1 and P_2 are proportional.

So distinct Poincaré series need not be independent. Petersson's formula, in the next section, measures how far they are from orthogonal.

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10.2. Petersson's formula

What is the Fourier expansion of P_m? For integers m, n and $c \geq 1$ let

$$S(m,n;c) := \sum_{\substack{d \bmod c \\ (d,c)=1}} e\Big(\frac{m\bar{d}+nd}{c}\Big), \qquad d\bar{d} \equiv 1 \pmod{c},$$

be the Kloosterman sum. It is real, and $S(m,n;c) = S(n,m;c)$.

Problem 10.2.1. The map $\gamma \mapsto (c,d)$ induces a bijection from $\Gamma_\infty\backslash\Gamma$ onto the set of pairs of coprime integers modulo ± 1. If $c \neq 0$, then $\gamma z = \frac{a}{c} - \frac{1}{c(cz+d)}$.

Problem 10.2.2. Let $\ell \geq 2$ be an integer, $\beta \in \mathbb{R}$ and $y > 0$. Then

$$\int_{\mathrm{i}y-\infty}^{\mathrm{i}y+\infty} z^{-\ell}\, e(-\beta z)\, dz = \begin{cases} (-2\pi\mathrm{i})^\ell\, \beta^{\ell-1}/(\ell-1)! & \beta > 0, \\ 0 & \beta \leq 0. \end{cases}$$

Problem 10.2.3. Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $y > 0$. Then $\int_{\mathrm{i}y-\infty}^{\mathrm{i}y+\infty} z^{-k}\, e(-\alpha/z - \beta z)\, dz$ vanishes for $\beta \leq 0$, and for $\beta > 0$ equals

$$2\pi\mathrm{i}^{-k}\Big(\frac{\beta}{\alpha}\Big)^{(k-1)/2} J_{k-1}\big(4\pi\sqrt{\alpha\beta}\big).$$

Problem 10.2.4. $P_m(z) = \sum_{n\geq 1} p_m(n)\, e(nz)$, where

$$p_m(n) = \delta(m,n) + 2\pi\mathrm{i}^{-k}\Big(\frac{n}{m}\Big)^{(k-1)/2} \sum_{c\geq 1} \frac{S(m,n;c)}{c}\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big).$$

Problems 10.1.1 and 10.2.4 together give Petersson's formula [Pe], [Iw, §3.2], [ILS, §2]. Let

$$T_k(m,n) := 2\pi\mathrm{i}^{-k} \sum_{c\geq 1} \frac{S(m,n;c)}{c}\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big),$$

a real symmetric function of $m, n \geq 1$.

Theorem 10.2.1 (Petersson)**.** *For every orthonormal basis (f_j) of S_k and all $m, n \geq 1$,*

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_j \overline{a_{f_j}(m)}\, a_{f_j}(n) = \delta(m,n) + T_k(m,n).$$

Problem 10.2.5. For $k \in \{4, 6, 8, 10, 14\}$ and all $m, n \geq 1$,

$$\sum_{c\geq 1} \frac{S(m,n;c)}{c}\, J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big) = -\frac{\mathrm{i}^k}{2\pi}\, \delta(m,n).$$

Numerically, for $m = n = 1$ the partial sums over $c \leq 800$ reproduce these values to within $3 \cdot 10^{-8}$ for $k = 4$ and 10^{-13} for $k \geq 6$. For $k = 12$ one finds $1 + T_{12}(1,1) = 2.84028\ldots$, which by Theorem 10.2.1 is $10!/\big((4\pi)^{11}\|\Delta\|^2\big)$, so that $\|\Delta\|^2 = 1.03536\ldots \cdot 10^{-6}$.

10.3. Near orthogonality

Normalize the Poincaré series by

$$\widetilde{P}_m := \frac{(4\pi m)^{(k-1)/2}}{\Gamma(k-1)^{1/2}}\, P_m,$$

and for each positive integer M let $G_M := \big(\langle \widetilde{P}_m, \widetilde{P}_n\rangle\big)_{1\le m,n\le M}$ be the $M \times M$ Gram matrix. We use the operator norm on $\mathbb{C}^M$ with its Euclidean inner product.

Problem 10.3.1. $\langle \widetilde{P}_m, \widetilde{P}_n\rangle = \delta(m,n) + T_k(m,n)$.

So G_M is the identity plus the matrix of Kloosterman terms, and the question is how small J_{k-1} is. We use Poisson's integral

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \int_{-1}^{1} (1-t^2)^{\nu-1/2} \cos(xt)\, dt \qquad (\nu > -\tfrac12,\ x>0),$$

see [DLMF, (10.9.4)].

Problem 10.3.2. For $\nu > -\frac12$ and $x > 0$, $|J_\nu(x)| \le (x/2)^\nu/\Gamma(\nu+1)$.

Problem 10.3.3. $|T_k(m,n)| \le 2\pi\, \zeta(k-1)\, (2\pi\sqrt{mn})^{k-1}/(k-1)!$.

Problem 10.3.4. If $1 \le M \le (k-1)/(4\pi e)$, then every eigenvalue of G_M lies within $k\,2^{-k}$ of 1.

Problems 10.3.1–10.3.4 prove the following.

Theorem 10.3.1. *If $1 \le M \le (k-1)/(4\pi e)$, then $\|G_M - 1\| \le k\,2^{-k}$. In particular $P_1, \dots, P_M$ are linearly independent, and $\dim S_k \ge \lfloor (k-1)/(4\pi e)\rfloor$.*

Here $e = 2.718\ldots$ is Euler's number, and $4\pi e = 34.15\ldots$. The threshold is not the true one. For $0 < x \le r\nu$ with $0 < r < 1$ fixed, the function $J_\nu(x)$ is exponentially small in ν, and Siegel's inequality makes this precise: for $\nu \ge 0$ and $0 < x \le 1$,

$$|J_\nu(\nu x)| \le \Big(\frac{x\, e^{\sqrt{1-x^2}}}{1+\sqrt{1-x^2}}\Big)^\nu$$

[DLMF, (10.14.5)].

Problem 10.3.5. Let $0 < \theta < 1/(4\pi)$. There are constants $0 < \varrho < 1$ and $C > 0$, depending only on θ, such that $\|G_M - 1\| \le Ck\varrho^k$ for every even $k \ge 4$ and every integer M with $1 \le M \le \theta(k-1)$.

So for large k the $\widetilde{P}_m$ with $m \le \theta(k-1)$, for a fixed $\theta < 1/(4\pi)$, are almost orthonormal. For m near $(k-1)/(4\pi)$ the argument $4\pi m$ of the term $c = 1$ of $T_k(m,m)$ reaches the transition region $x \approx \nu$ of J_ν, where $J_\nu(x)$ is no longer exponentially small but of size up to about $\nu^{-1/3}$, and the diagonal entry is in general no longer exponentially close to 1. For $k = 100$, where $(k-1)/(4\pi) = 7.87\ldots$ and $\dim S_{100} = 8$, a computation of the Kloosterman terms gives the following. The eigenvalues of G_7 lie in $[0.99999, 1.0094]$. Those of G_8 lie in $[0.982, 1.808]$, so $P_1, \dots, P_8$ form a basis of S_{100}, and the Gram matrix of the normalized basis $\widetilde{P}_1, \dots, \widetilde{P}_8$ has an eigenvalue near 1.808. G_9 is singular, as it must be.

10.4. The dimension as formal degree times covolume

How large a proportion of S_k does Theorem 10.3.1 fill? Let $\Gamma\backslash\mathbb{H}$ carry the measure ν.

Problem 10.4.1. $\nu(\Gamma\backslash\mathbb{H}) = \pi/3$.

Problem 10.4.2. $\left|\dim S_k - \frac{k-1}{12}\right| \le \frac{13}{12}$ for every even $k \ge 4$.

For fixed $0 < \theta < 1/(4\pi)$, the first $\lfloor\theta(k-1)\rfloor$ series in Problem 10.3.5 span a proportion of S_k tending to 12θ as k tends to infinity through even weights. These limits approach $3/\pi = 0.954\ldots$ as θ increases to $1/(4\pi)$. The full initial segment in Theorem 10.3.1 gives limiting proportion $3/(\pi e) = 0.351\ldots$. Where does $(k-1)/12$ come from? It is the product of $\pi/3$ and a constant attached to the weight alone.

Let $G := \mathrm{PSL}_2(\mathbb{R})$ act on $\mathbb{H}$, and normalize Haar measure dg by $\int_G F(g\mathrm{i})\,dg = \int_{\mathbb{H}} F\,d\nu$ for compactly supported continuous F. With Haar probability on the stabilizer $\mathrm{PSO}(2)$ in the quotient integral formula, $\mathrm{vol}(\mathrm{PSL}_2(\mathbb{Z})\backslash G) = \pi/3$. Let $\mathcal{A}_k$ be the space of holomorphic functions f on $\mathbb{H}$ with $\|f\|^2 := \int_{\mathbb{H}} |f|^2 y^k\,d\nu < \infty$, on which G acts unitarily by $(\pi(g)f)(z) := f(g^{-1}z)\,j(g^{-1},z)^{-k}$. This is the holomorphic discrete series of weight k. For $w \in \mathbb{H}$ put

$$h_w(z) := \Big(\frac{z-\bar{w}}{2\mathrm{i}}\Big)^{-k}.$$

For a unit vector u of an irreducible unitary representation π with square-integrable matrix coefficients, the *formal degree* d_π is defined by $\int_G |\langle\pi(g)u,u\rangle|^2\,dg = 1/d_\pi$. It does not depend on u, by the Schur orthogonality relations for such representations [HC]. We compute it with u proportional to h_{i}.

Problem 10.4.3. For every $w \in \mathbb{H}$, $h_w \in \mathcal{A}_k$. For every $f \in \mathcal{A}_k$, $\langle f, h_{\mathrm{i}}\rangle = \frac{4\pi}{k-1} f(\mathrm{i})$.

Problem 10.4.4. $\pi(g)h_w = \overline{j(g,w)}^{\,-k} h_{gw}$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathbb{H}$.

Problem 10.4.5. Let $u := h_{\mathrm{i}}/\|h_{\mathrm{i}}\|$, and for $g \in G$ write $w := g\mathrm{i}$. Then $|\langle\pi(g)u,u\rangle|^2 = \big(4\,\mathrm{Im}\,w/|w+\mathrm{i}|^2\big)^k$, and the formal degree of π is $(k-1)/(4\pi)$.

With Problem 10.4.1 this gives

$$d_\pi \cdot \mathrm{vol}(\mathrm{PSL}_2(\mathbb{Z})\backslash G) = \frac{k-1}{4\pi}\cdot\frac{\pi}{3} = \frac{k-1}{12},$$

which is $\dim S_k$ up to the bounded error of Problem 10.4.2. This is not a coincidence. As in Problem 10.1.1, and using Problems 10.4.3 and 10.4.4, one checks that the reproducing kernel $B(z,w) := \sum_j f_j(z)\overline{f_j(w)}$ of S_k, for an orthonormal basis (f_j), is

$$B(z,w) = \frac{k-1}{4\pi}\sum_{\gamma\in\mathrm{PSL}_2(\mathbb{Z})} j(\gamma,z)^{-k}\,h_w(\gamma z).$$

Since $\dim S_k = \int_{\Gamma\backslash\mathbb{H}} B(z,z)\,y^k\,d\nu(z)$, the term $\gamma = 1$ contributes exactly the formal degree times the covolume, and by Problem 10.4.2 all other terms together contribute a bounded amount. For a compact unitary group with Haar probability, the formal degree of an irreducible representation is its dimension. In the definite example of Chapter 21, for rank $n \ge 1$ and weight λ with $4 \mid |\lambda|$, the central scalars contribute $4M_n \dim V_\lambda$, where M_n is the mass (Problem 21.3.4).

Theorem 10.3.1 says that the map $\mathbb{C}^M \to S_k$, $e_m \mapsto \widetilde{P}_m$, is almost an isometry when M is at most a fixed proportion of $\dim S_k$. Its image therefore occupies only part of S_k. Problem 10.3.5 extends near orthonormality to a longer initial segment, and Problem 10.1.2 shows that the full family spans S_k. In the setting of Chapter 9, a finite group acts on the ambient space. A near-isometry estimate for the projected map and a second-moment bound for the Gram operators of the unprojected map and its translates give a lower bound for the dimension spanned by a few projected translates in the target (Theorems 9.3.1 and 9.4.1).

The analogue in the proof. This chapter gives a classical model for the estimates in Chapter 36. In the proof, the principal-level forms on the unitary group of a hermitian space Y_0 over a CM field E play the role of S_k. Bessel periods play the role of Fourier coefficient extraction and are represented by a Poincaré map. Its values on the source coordinate basis are Bessel–Poincaré series. The two estimates below concern the map followed by the projector onto the good forms and the dimension of those forms. The next paragraph gives the declarations for these spaces, the period map and its values on the basis, and the two estimates. For the definition of Bessel periods see [GGP, §§12, 23].

In the Lean formalization. Fix admissible data D (`New.Data`) of the Lean proof [GL]. Write E for `Data.E`, Y_0 for `D.spaceY 0`, and r for the rank of Y_0 over E. Write S for the source forms `Data.SourceForm`, with coordinates `Data.sourceFormCoords`, and D_X for their complex dimension. The Poincaré map `Data.poincare` is `Data.poincareAt` at the chosen archimedean point `Data.besselPoint` and linear form `Data.besselForm`. By `Data.inner_poincareAt` it represents the Bessel period for each source form and each principal-level test form. By `Data.poincare_eq_poincareMap_comp` and `Filling.Fits.coe_poincareMap`, the inverse coordinate images of the standard basis vectors map to the corresponding Bessel–Poincaré series.

Let P be `Data.scaledPoincare`, the Poincaré map in these Euclidean coordinates multiplied by the positive factor `Data.poincareScale`. Its codomain is the space `Data.principalForms` on Y_0, of the specified discrete-series type at infinity and fixed by the kernel of reduction at the chosen finite places inside the standard finite level. Let e_0 denote the action of the group-algebra idempotent $\texttt{D.e}_0$ through `Data.principalRep` on this space. It projects onto the space T of good forms (`Data.targetForms`), of complex dimension d_T (`Data.targetDim`). Let b be the weight `D.b` at the infinite places `D.bPlaces`. For every complex subspace U of the source coordinate space, $\texttt{Data.isNearIsometry_e}_0\texttt{_comp_scaledPoincare}$ gives

$$\|e_0 P \operatorname{pr}_U\|_{\mathrm{HS}}^2 \geq \dim U - 5b^{-4} D_X,$$

where pr_U is the orthogonal projection and $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm.

By `Data.targetDim_le_card_mul_besselDegree_mul_mass`,

$$d_T \leq \#\bar{G} \cdot \deg \cdot \operatorname{mass}(E, r).$$

Here $\bar{G}$ is `Data.chosenGL`, the product of GL_r over the residue fields at the chosen places above the two level primes. The constant deg is `D.filling.besselDegree`, the product of the degrees of the local Bessel systems over all infinite places of E. The factor $\operatorname{mass}(E, r)$ is `New.mass`, evaluated at E and r, the mass of the genus of the standard hermitian lattice $\mathcal{O}_E^r$. Both inequalities are in `grh/src/New/Spanning/SecondMoment/SpanningTheorem.lean`.

Hints

§10.1.

10.1.1: Unfold the sum over $\Gamma_\infty\backslash\Gamma$ to the strip $0 \le x \le 1$, $y > 0$, and integrate in x first.

10.1.2: Problem 10.1.1.

10.1.3: Expand P_m in the basis Δ and use Problem 10.1.1. Note that $\tau(1) = 1$, $\tau(2) = -24$.

§10.2.

10.2.1: The bottom row of a matrix in Γ is coprime, and two matrices with the same bottom row differ by left multiplication by an element of Γ_∞.

10.2.2: For $\beta \neq 0$, close the contour in the half-plane where $e(-\beta z)$ decays. For $\beta = 0$, use a primitive of $z^{-\ell}$.

10.2.3: Expand $e(-\alpha/z)$ in its power series and apply Problem 10.2.2 term by term. For $\beta \le 0$, $|e(-\alpha/z)| \le 1$ in $\mathbb{H}$.

10.2.4: The terms with $c > 0$ and fixed c are indexed by $d \in \mathbb{Z}$ coprime to c; group them by d mod c. Each group is a sum over translates $z \mapsto z + r$, $r \in \mathbb{Z}$, so its n-th Fourier coefficient is an integral over a whole horizontal line. Use Problems 10.2.1 and 10.2.3.

10.2.5: Theorem 10.2.1.

§10.3.

10.3.1: Problem 10.1.1 with $f = P_m$, and Problem 10.2.4.

10.3.2: Bound $|\cos(xt)|$ by 1 and evaluate the Beta integral.

10.3.3: $|S(m, n; c)| \le c$.

10.3.4: $(k-1)! \ge ((k-1)/e)^{k-1}$. Then Gershgorin's theorem.

10.3.5: Treat the term $c = 1$ with Siegel's inequality. The base on its right is increasing in x and equals 1 at $x = 1$. For the terms $c \ge 2$, use Problem 10.3.2.

§10.4.

10.4.1: The standard fundamental domain $|x| \le \frac{1}{2}$, $|z| \ge 1$; integrate in y first.

10.4.2: Write $k = 12j + r$ with $0 \le r < 12$ and use the dimension formula.

10.4.3: Put $\zeta := (z - \mathrm{i})/(z + \mathrm{i})$ and $F(\zeta) := f(z)\big((z + \mathrm{i})/2\mathrm{i}\big)^k$. Express $|f|^2 y^k \, d\nu$ in terms of F and ζ, and use the mean value property on the circles $|\zeta| = r$. For $h_w \in \mathcal{A}_k$, use $4 \operatorname{Im} z \operatorname{Im} w/|z - \bar{w}|^2 = 1 - |\zeta_w|^2$ with $\zeta_w := (z - w)/(z - \bar{w})$.

10.4.4: $gz - \overline{gw} = (z - \bar{w})/\big(j(g, z)\overline{j(g, w)}\big)$.

10.4.5: Problems 10.4.3 and 10.4.4. For the integral, use geodesic polar coordinates about i, in which $4 \operatorname{Im} w/|w + \mathrm{i}|^2 = \cosh^{-2}(\rho/2)$.

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