FEW TRANSLATES SPAN

Prerequisites. Finite-dimensional complex inner product spaces: adjoints, orthogonal projections, the spectral theorem for self-adjoint maps, and the trace. No earlier chapter is used.

In the proof, Theorem B bounds how many translates of a map between spaces of automorphic forms are needed for their images to span almost all of the target. The translates are indexed by a finite group. When do a few translates of one map span? Section 9.1 bounds the rank of a sum of images from below by a second moment. The rest of the chapter turns a second moment over the group into a choice of few translates, by Cauchy–Schwarz and a greedy choice. All spaces are finite-dimensional, and inner products are linear in the second variable.

9.1. Gram matrices

For a linear map A between inner product spaces, $\|A\|_{\mathrm{HS}}^2 := \operatorname{tr}(A^*A)$ is the squared Hilbert–Schmidt norm. Let $\mathcal{H}$ be a complex inner product space of dimension $D' > 0$. Let $B_i \colon X_i \to \mathcal{H}$, $i \in I$, be a finite family of linear maps from inner product spaces of dimensions r_i, with $R := \sum_i r_i > 0$, let $\gamma_i > 0$, and let $K_i := \gamma_i B_i^* B_i$ and $Q := \sum_i \gamma_i B_i B_i^*$. Let $X := \bigoplus_i X_i$ and $\Phi \colon X \to \mathcal{H}$, $\Phi((x_i)_i) := \sum_i \sqrt{\gamma_i} B_i x_i$, so that $Q = \Phi\Phi^*$. The *Gram matrix* of the family is $G := \Phi^*\Phi$. Its blocks are $G_{ij} = \sqrt{\gamma_i\gamma_j}\, B_i^* B_j \colon X_j \to X_i$, and $G_{ii} = K_i$. When every X_i is a line spanned by a unit vector ξ_i, G is the Gram matrix of the vectors $\sqrt{\gamma_i} B_i \xi_i$.

Problem 9.1.1. If $Q \neq 0$, then $\operatorname{rank} Q \geq (\operatorname{tr} Q)^2 / \operatorname{tr} Q^2$, with equality exactly when Q is a multiple of an orthogonal projection.

Problem 9.1.2. $Q(\mathcal{H}) = \sum_i B_i(X_i)$.

Problem 9.1.3. Q and G have the same nonzero eigenvalues, with multiplicities. Hence $\operatorname{tr} Q = \sum_i \operatorname{tr} K_i$ and $\operatorname{tr} Q^2 = \sum_{i,j} \|G_{ij}\|_{\mathrm{HS}}^2$.

Problems 9.1.1–9.1.3 prove the following.

Theorem 9.1.1. *If some B_i is nonzero, then*

$$\dim \sum_i B_i(X_i) \geq \frac{\left(\sum_i \operatorname{tr} K_i\right)^2}{\sum_{i,j} \|G_{ij}\|_{\mathrm{HS}}^2}.$$

For lines this is a form of Welch's bound [We]. Unlike the left side, the right side changes little when the B_i are perturbed.

Problem 9.1.4. Let $v, w \in \mathbb{C}^2$ be unit vectors, $X_1 = X_2 = \mathbb{C}$, $B_1(1) = v$, $B_2(1) = w$ and $\gamma_1 = \gamma_2 = 1$. The right side of Theorem 9.1.1 is $2/(1 + |\langle v, w\rangle|^2)$.

So two nearly parallel lines count as about one direction. In the proof a finite group acts on the target of one map B. The moment hypothesis bounds the dimension of the projected target times the sum of the squared overlaps $\|B^*\rho(s)B\|_{\mathrm{HS}}^2$ over all group elements, including the identity (Theorem 9.3.1 below).

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9.2. Translates and the second moment

From now on S and T are finite-dimensional complex inner product spaces, Γ is a finite group, and ρ is a representation of Γ on T with $\langle \rho(s)x, \rho(s)y\rangle = \langle x, y\rangle$ for all $s \in \Gamma$ and $x, y \in T$. Let $B\colon S \to T$ be linear. For a subspace $W \subseteq T$ let P_W be the orthogonal projection of T onto W, as a map $T \to T$. The *second moment* of B is $\sum_{s\in\Gamma} \|B^*\rho(s)B\|_{\mathrm{HS}}^2$. Write $\langle A, A'\rangle_{\mathrm{HS}} := \operatorname{tr}(A^*A')$ for linear maps A, A' with the same source and target.

Problem 9.2.1. For every orthonormal basis (e_k) of S and all $A, A'\colon S \to T$, $\langle A, A'\rangle_{\mathrm{HS}} = \sum_k \langle Ae_k, A'e_k\rangle$. For every subspace $W \subseteq T$, $\|P_W\|_{\mathrm{HS}}^2 = \dim W$ and $\|P_W A\|_{\mathrm{HS}}^2 = \langle P_W, AA^*\rangle_{\mathrm{HS}}$.

Consider the family of all translates, $I = \Gamma$, $X_s = S$, $B_s = \rho(s)B$ and $\gamma_s = 1$, in the notation of Section 9.1.

Problem 9.2.2. $G_{st} = B^*\rho(s^{-1}t)B$, and $\langle \rho(s)BB^*\rho(s)^*, \rho(t)BB^*\rho(t)^*\rangle_{\mathrm{HS}} = \|B^*\rho(s^{-1}t)B\|_{\mathrm{HS}}^2$. Hence

$$\Big\|\sum_{s\in\Gamma} \rho(s)BB^*\rho(s)^*\Big\|_{\mathrm{HS}}^2 = \sum_{s,t\in\Gamma} \|G_{st}\|_{\mathrm{HS}}^2 = \#\Gamma \sum_{s\in\Gamma} \|B^*\rho(s)B\|_{\mathrm{HS}}^2.$$

Problem 9.2.3. Let $S \neq 0$, let B be an isometry, and let $\varepsilon \geq 0$. If $\dim T \cdot \sum_s \|B^*\rho(s)B\|_{\mathrm{HS}}^2 \leq \#\Gamma\,(1+\varepsilon)(\dim S)^2$, then the images $\rho(s)B(S)$, $s \in \Gamma$, together span a subspace of dimension at least $\dim T/(1+\varepsilon)$.

Problem 9.2.3 uses all $\#\Gamma$ translates. For $0 < \varepsilon < 1/16$, Theorem 9.4.1 will give a span of at least $(1-16\varepsilon)\dim T$ dimensions in this setting, using at most $\frac{5}{2}(1 + \log(1/(16\varepsilon)))\dim T/\dim S$ translates. The first step is a bound for one subspace at a time.

Problem 9.2.4. For every subspace $W \subseteq T$,

$$\Big(\sum_{s\in\Gamma} \|P_W\rho(s)B\|_{\mathrm{HS}}^2\Big)^2 \leq \dim W \cdot \#\Gamma \sum_{s\in\Gamma} \|B^*\rho(s)B\|_{\mathrm{HS}}^2.$$

An isometry condition on the projected map gives a lower bound for the second moment. Let $p\colon T \to T$ be a linear map with $p^2 = p$ and $p^* = p$, commuting with every $\rho(s)$, and let $T_0 := \operatorname{im} p$.

Problem 9.2.5. T_0 and $T_0^\perp$ are stable under Γ, and $p = P_{T_0}$. If pB is an isometry, then $\#\Gamma\,(\dim S)^2 \leq \dim T_0 \cdot \sum_s \|B^*\rho(s)B\|_{\mathrm{HS}}^2$.

When pB is an isometry and $T_0 \neq 0$, Problem 9.2.5 says that the mean of $\|B^*\rho(s)B\|_{\mathrm{HS}}^2$ over $s \in \Gamma$ is at least $(\dim S)^2/\dim T_0$. The hypothesis below requires it to be at most $1+\varepsilon$ times this value.

9.3. One translate

Let $A\colon S \to T$ be linear and $\varepsilon \in \mathbb{R}$. We say that A is a *near isometry with error* ε if $\dim U - \varepsilon \dim S \leq \|AP_U\|_{\mathrm{HS}}^2$ for every subspace $U \subseteq S$, where now $P_U\colon S \to S$ is the orthogonal projection onto U. This is a lower bound in mean square only.

Problem 9.3.1. If $\varepsilon \geq 0$ and $\|A^*A - 1\|_{\mathrm{HS}}^2 \leq \varepsilon^2 \dim S$, then A is a near isometry with error ε.

Let Θ be a set, let V and Y be finite-dimensional complex vector spaces, and let $C_\theta\colon V \to Y$, $\theta \in \Theta$, be linear maps. For $\eta \in \mathbb{R}$, we say that the family (C_θ) *spreads with error* η if for every subspace $F \subseteq Y$, with $\alpha := \dim F / \dim Y$ (and $\alpha := 0$ if $Y = 0$), there is $\theta \in \Theta$ with

$$\dim C_\theta^{-1}(F) \le \Big(\frac{1+\alpha}{2} + \eta\Big) \dim V.$$

The index θ may depend on F.

Now return to Section 9.2, and let $C_s := p\rho(s)B$, viewed as a map $S \to T_0$. Assume that $\varepsilon \ge 0$, that pB is a near isometry with error ε, and that

$$\dim T_0 \cdot \sum_{s\in\Gamma} \|B^*\rho(s)B\|_{\mathrm{HS}}^2 \le \#\Gamma\,(1+\varepsilon)(\dim S)^2. \tag{9.3.1}$$

Let $W \subseteq T_0$ be a subspace and let $\alpha := \dim W / \dim T_0$, with $\alpha := 0$ if $T_0 = 0$.

Problem 9.3.2. Some $s \in \Gamma$ has $\|P_W\rho(s)B\|_{\mathrm{HS}}^2 \le \frac{1}{2}(1+\alpha+\varepsilon)\dim S$.

Problem 9.3.3. For every $s \in \Gamma$, let $U := C_s^{-1}(W) \subseteq S$. Then $\|pBP_U\|_{\mathrm{HS}}^2 \le \|P_W\rho(s)B\|_{\mathrm{HS}}^2$. For s as in Problem 9.3.2, $\dim U \le \big(\frac{1+\alpha}{2} + \frac{3}{2}\varepsilon\big)\dim S$.

Problems 9.2.4, 9.3.2 and 9.3.3 prove the following, which is a step of the proof.

Theorem 9.3.1. *Let S and T be finite-dimensional complex inner product spaces, Γ a finite group, and ρ a representation of Γ on T with $\langle\rho(s)x, \rho(s)y\rangle = \langle x, y\rangle$ for all $s \in \Gamma$ and $x, y \in T$. Let $p\colon T \to T$ be linear with $p^2 = p$, $p^* = p$ and $p\rho(s) = \rho(s)p$ for every $s \in \Gamma$. Let $B\colon S \to T$ be linear and $\varepsilon \in \mathbb{R}$, with $\varepsilon \ge 0$, such that pB is a near isometry with error ε and* (9.3.1) *holds with $T_0 := \operatorname{im} p$. Then the maps $p\rho(s)B\colon S \to T_0$, $s \in \Gamma$, spread with error $\frac{3}{2}\varepsilon$.*

Here $T_0 = 0$ and $\varepsilon = 0$ are allowed.

9.4. The greedy choice

Return to maps $C_\theta\colon V \to Y$, $\theta \in \Theta$, as in Section 9.3. For a subspace $F \subseteq Y$ let $c(F) := \dim Y - \dim F$ be its codimension. In Problems 9.4.2–9.4.4 assume that $\dim Y > 0$, that $0 < \varepsilon < 1/16$, and that the family spreads with error $\frac{3}{2}\varepsilon$, and put $c_0 := 2\dim V/(5\dim Y)$.

Problem 9.4.1. For every linear $A\colon V \to Y$ and every subspace $F \subseteq Y$, $\dim(F + A(V)) + \dim A^{-1}(F) = \dim F + \dim V$.

Problem 9.4.2. If $c(F) > 16\varepsilon \dim Y$, then some θ has $c(F + C_\theta(V)) \le (1-c_0)\,c(F)$.

Problem 9.4.3. For every integer $\ell \ge 0$ there is a set $t \subseteq \Theta$ with $\#t \le \ell$ and $c\big(\sum_{\theta\in t} C_\theta(V)\big) \le \max\big(16\varepsilon, e^{-\ell c_0}\big)\dim Y$.

Problem 9.4.4. If $V \ne 0$, then $0 < c_0 \le 1$.

Problem 9.4.5. Under the hypotheses of Problem 9.4.4, with $V \ne 0$, there is an integer $\ell \ge 0$ such that $e^{-\ell c_0} \le 16\varepsilon$ and $\ell c_0 \le 1 + \log(1/(16\varepsilon))$, where log is the natural logarithm.

Problems 9.4.3 and 9.4.5 give the following theorem, used in the proof.

Theorem 9.4.1. *Let Θ be a set, let V and Y be finite-dimensional complex vector spaces, let $C_\theta\colon V \to Y$, $\theta \in \Theta$, be linear maps, and let $\varepsilon \in \mathbb{R}$. Assume that $V \neq 0$, that the family spreads with error $\frac{3}{2}\varepsilon$, that $\dim Y > 0$, and that $0 < \varepsilon < 1/16$. Then there is a finite set $t \subseteq \Theta$ with*

$$\#t \cdot \dim V \le \frac{5}{2}\Big(1 + \log\frac{1}{16\varepsilon}\Big)\dim Y \quad \textit{and} \quad (1-16\varepsilon)\dim Y \le \dim \sum_{\theta\in t} C_\theta(V).$$

Applied with $\Theta = \Gamma$, $V = S$, $Y = T_0$ and $C_s = p\rho(s)B$, Theorems 9.3.1 and 9.4.1 say: if $S \neq 0$, $T_0 \neq 0$, $0 < \varepsilon < 1/16$, pB is a near isometry with error ε and (9.3.1) holds, then the images of at most $\frac{5}{2}(1 + \log(1/(16\varepsilon)))\dim T_0/\dim S$ of the maps $p\rho(s)B$ span at least $(1-16\varepsilon)\dim T_0$ dimensions of T_0. This is the step that Theorem B of the proof needs. There (Chapter 36) Γ is the finite group $\bar{G}$, p is the action of the good idempotent e_0, T_0 is the space T of target forms, of dimension d_T, and the theorems are used with $\varepsilon = 1/(8b^2)$, where b is the weight at the b-places (Chapter 33). The resulting statement, after restriction to the ramified source forms, is Theorem 36.4.2. For a related logarithmic bound in greedy submodular covering, see Wolsey [Wo].

In the Lean formalization. In the Lean proof [GL], file `grh/src/New/Spanning/SecondMoment/FewTranslates.l` Theorem 9.3.1 is `New.isSpread_of_second_moment` and Theorem 9.4.1 is `New.IsSpread.exists_few_le_finrank_iSup_range`; spreading and near isometry are `New.IsSpread` and `New.IsNearIsometry`.

9.5. Translations of a cyclic group

Let $\Gamma = \mathbb{Z}/N\mathbb{Z}$, $T = \mathbb{C}^\Gamma$ with $\langle f, g\rangle = \sum_x \overline{f(x)}g(x)$, $(\rho(s)f)(x) = f(x-s)$, and $1 \le M \le N$. Let $S = \mathbb{C}^M$ with standard basis $e_0, \dots, e_{M-1}$, and $I := \{0, 1, \dots, M-1\} \subseteq \mathbb{Z}/N\mathbb{Z}$.

Problem 9.5.1. Let $Be_j := \delta_j$, the indicator function of $\{j\}$. Then $\|B^*\rho(s)B\|_{\mathrm{HS}}^2 = \#(I \cap (I+s))$, and the second moment is M^2. Hence, with $p = 1$, (9.3.1) holds with $\varepsilon = 0$, with equality.

Problem 9.5.2. In Problem 9.5.1, for $0 < \varepsilon < 1/16$, the least number of translates $\rho(s)B$ whose images span at least $(1-16\varepsilon)N$ dimensions is $\lceil (1-16\varepsilon)N/M \rceil$. For $N = 12$, $M = 3$, $\varepsilon = 1/128$ it is 4, while the bound of Theorem 9.4.1 is $10(1+\log 8) \approx 30.79$.

Every translate of $B(S)$ is orthogonal to any Γ-stable subspace orthogonal to $B(S)$.

Problem 9.5.3. Let $Be_j(x) := N^{-1/2}e^{2\pi i jx/N}$, $0 \le j < M$. Then B is an isometry, every $B^*\rho(s)B$ is diagonal and unitary, the second moment is NM, and the images of any nonempty set of translates span exactly $B(S)$. With $p = 1$, (9.3.1) holds exactly when $N \le (1+\varepsilon)M$. The subspace $B(S)$ is Γ-stable, and with $p = P_{B(S)}$, (9.3.1) holds with $\varepsilon = 0$.

Problem 9.5.4. In the setting of Section 9.2, let $S \neq 0$, let B be an isometry, and let $T_1 \subseteq T$ be a Γ-stable subspace orthogonal to $B(S)$. If (9.3.1) holds with $p = 1$, then $\dim T_1 \le \varepsilon \dim T/(1+\varepsilon)$.

Thus a stable subspace orthogonal to $B(S)$ gives a lower bound for the second moment. In general the theorems bound the dimension spanned inside the chosen target $T_0 = \operatorname{im} p$.

Hints

§9.1.

9.1.1: $Q = \Phi\Phi^* \geq 0$. Cauchy–Schwarz for the nonzero eigenvalues of Q.
9.1.2: $\ker \Phi^* = (\operatorname{im} \Phi)^\perp$.
9.1.3: If $Gx = \lambda x$ with $\lambda \neq 0$, then $Q(\Phi x) = \lambda \Phi x$. Then expand $\operatorname{tr} G^2$ by blocks.
9.1.4: G is the 2×2 matrix with diagonal entries 1 and off-diagonal entries $\langle v, w\rangle$ and $\langle w, v\rangle$.

§9.2.

9.2.1: For the last identity, $\operatorname{tr}(XY) = \operatorname{tr}(YX)$ and $P_W^2 = P_W = P_W^*$.
9.2.2: $\rho(s)^* = \rho(s^{-1})$; then Problem 9.1.3, and count the pairs (s, t) with given $s^{-1}t$.
9.2.3: Theorem 9.1.1 and Problem 9.2.2.
9.2.4: Sum Problem 9.2.1 over s, apply Cauchy–Schwarz for $\langle \cdot, \cdot \rangle_{\mathrm{HS}}$, and use Problem 9.2.2.
9.2.5: Problem 9.2.4 with $W = T_0$.

§9.3.

9.3.1: $\|AP_U\|_{\mathrm{HS}}^2 = \dim U + \langle P_U, A^*A - 1\rangle_{\mathrm{HS}}$.
9.3.2: Problem 9.2.4, with W viewed inside T, and $\sqrt{ab} \leq (a + b)/2$.
9.3.3: Compare $\rho(s)pB$ and $P_W \rho(s)B$ on U.

§9.4.

9.4.1: Rank–nullity for the restriction of A to $A^{-1}(F)$, and $\dim(F + A(V)) + \dim(F \cap A(V)) = \dim F + \dim A(V)$.
9.4.2: Problem 9.4.1 with $A = C_\theta$. Compare $\frac{1+\alpha}{2} + \frac{3}{2}\varepsilon$ with $1 - \frac{2}{5}(1 - \alpha)$.
9.4.3: Induction on ℓ, and $1 - c_0 \leq e^{-c_0}$.
9.4.4: Problem 9.4.2 with $F = 0$.
9.4.5: Round $\log(1/(16\varepsilon))/c_0$ up to an integer.

§9.5.

9.5.1: Count the pairs $(j, k) \in I \times I$ with $j - k = s$.
9.5.2: The images are spanned by indicator functions of points.
9.5.3: Each Be_j is an eigenvector of every $\rho(s)$.
9.5.4: Problem 9.2.5 with $p = P_{T_1^\perp}$.

References

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