THE FOCK MODEL AND THETA INTERTWINERS

Prerequisites. Fourier series of entire periodic functions. Quadratic Gauss sums: for an odd prime p and $p \nmid a$, $\sum_{t \bmod p} e^{2\pi i a t^2/p} = \left(\frac{a}{p}\right)\varepsilon_p\sqrt{p}$, with $\varepsilon_p = 1$ for $p \equiv 1 \pmod 4$ and $\varepsilon_p = \mathrm{i}$ for $p \equiv 3 \pmod 4$ [Dav, Ch. 2]. From Chapter 11, *Complex tori, theta functions and periods*: Jacobi's function and its quasi-periodicity (Section 11.2, Problem 11.2.1).

The kernels of the definite theta lifts of later chapters are built from theta functions of lattices of homomorphisms. Here we ask how a theta function of one lattice becomes a theta function of another, and what is lost when one does this twice. For compatible lattices a finite sum of translation operators does this, and for the neighbours below the loss is a root of unity given by a Gauss sum. We work in one complex variable, with $\mathbb{Z}[\mathrm{i}]$ and its neighbours of index 2 and p.

12.1. The Heisenberg group and the Fock model

On $V = \mathbb{C}$ let $H(z,w) := z\bar{w}$ and $E := \operatorname{Im} H$, so that $E(x+\mathrm{i}y, x'+\mathrm{i}y') = yx' - xy'$ is a symplectic form. For $x \in \mathbb{C}$ and a function G on $\mathbb{C}$ put

$$(U_x G)(z) := G(z+x)\, e^{-\pi H(z,x) - \frac{\pi}{2}H(x,x)}.$$

Let $\mathcal{F}$ be the *Fock space* of entire functions with $\|G\|^2 := \int_{\mathbb{C}} |G(z)|^2 e^{-\pi|z|^2}\, dA(z) < \infty$. The operators $e^{\mathrm{i}s}U_x$ with $s \in \mathbb{R}$ form a Heisenberg group, and $\mathcal{F}$ is its Fock model [Fo, Ch. 1], [Pol, Ch. 2]; for theta functions in one variable from this point of view, see [Mu, Ch. I, §3].

Problem 12.1.1. Each U_x is a unitary operator on $\mathcal{F}$, and

$$U_x U_y = e^{-\pi \mathrm{i} E(x,y)}\, U_{x+y}.$$

Problem 12.1.2. The monomials z^n are orthogonal in $\mathcal{F}$, with $\|z^n\|^2 = n!/\pi^n$.

So $f \mapsto f(\sqrt{\pi}\, z)$ is an isometry from $\mathbb{C}[z]$ with the Gaussian norm $\|f\|^2 := \pi^{-1}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dA(z)$, for which $\|z^n\|^2 = n!$, into $\mathcal{F}$.

The representation of the Heisenberg group on $\mathcal{F}$ is irreducible. By the Stone–von Neumann theorem, every strongly continuous irreducible unitary representation on which $e^{\mathrm{i}s}$ acts by $e^{\mathrm{i}s}$ is unitarily equivalent to it [LV], [Pol, Ch. 2]. The inverse image of a self-dual lattice in the Heisenberg group is maximal abelian. A theta function is a joint eigenfunction for this group; the centre acts by its given character and the lattice eigenvalues are specified by a semicharacter.

A lattice $\Lambda \subset \mathbb{C}$ is *self-dual* if $E(\Lambda, \Lambda) \subset \mathbb{Z}$ and $\mathbb{C}/\Lambda$ has area 1. A *semicharacter* of Λ is a function $\alpha\colon \Lambda \to \{\pm 1\}$ with

$$\alpha(\lambda+\mu) = \alpha(\lambda)\alpha(\mu)(-1)^{E(\lambda,\mu)} \qquad (\lambda, \mu \in \Lambda).$$

For such a pair let $\mathcal{T}(\Lambda, \alpha)$ be the space of entire functions G with $\alpha(\lambda)U_\lambda G = G$ for all $\lambda \in \Lambda$, that is,

$$G(z+\lambda) = \alpha(\lambda)\, e^{\pi z\bar{\lambda} + \frac{\pi}{2}|\lambda|^2}\, G(z).$$

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These are the classical theta functions of the line bundle on $\mathbb{C}/\Lambda$ with Hermitian form H and semicharacter α [Pol, Ch. 1]. Let $\alpha_0(m+n\mathrm{i}) := (-1)^{mn}$ on $\mathbb{Z}[\mathrm{i}]$.

Problem 12.1.3. The semicharacters of $\mathbb{Z}[\mathrm{i}]$ are the four functions $(-1)^{mn+am+bn}$ with $a, b \in \{0, 1\}$. Exactly two of them, α_0 and $(-1)^{mn+m+n}$, are invariant under $v \mapsto \mathrm{i}v$.

Problem 12.1.4. Let α be a semicharacter of a self-dual Λ. Then $\lambda \mapsto \alpha(\lambda)U_\lambda$ is a homomorphism, and for $w \in \mathbb{C}$ and $G \in \mathcal{T}(\Lambda, \alpha)$,

$$\alpha(\lambda)U_\lambda\,(U_w G) = e^{-2\pi\mathrm{i}E(\lambda,w)}\,U_w G \qquad (\lambda \in \Lambda).$$

Problem 12.1.5. For $G \in \mathcal{T}(\Lambda, \alpha)$ the function $z \mapsto |G(z)|^2 e^{-\pi|z|^2}$ is Λ-periodic. Moreover $\mathcal{T}(\Lambda, \alpha) \cap \mathcal{F} = 0$.

So no nonzero theta function lies in $\mathcal{F}$. For $G \in \mathcal{T}(\Lambda, \alpha)$ we use the norm

$$\|G\|_\Lambda^2 := \int_{\mathbb{C}/\Lambda} |G(z)|^2 e^{-\pi|z|^2}\,dA(z).$$

12.2. Theta functions of $\mathbb{Z}[\mathrm{i}]$

How many theta functions does $\mathbb{Z}[\mathrm{i}]$ carry? For $\operatorname{Im}\tau > 0$ let $\vartheta_\tau(z) := \sum_{n\in\mathbb{Z}} e^{\pi\mathrm{i}n^2\tau + 2\pi\mathrm{i}nz}$, the function ϑ of Chapter 11, and let

$$G(z) := e^{\pi z^2/2}\,\vartheta_{\mathrm{i}}(z) = e^{\pi z^2/2}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 + 2\pi\mathrm{i}nz}.$$

Problem 12.2.1. $\mathcal{T}(\mathbb{Z}[\mathrm{i}], \alpha_0) = \mathbb{C}G$.

Problem 12.2.2. $G(\mathrm{i}z) = G(z)$.

Problem 12.2.3. $\|G\|^2_{\mathbb{Z}[\mathrm{i}]} = 2^{-1/2}$.

For $t > 0$ let $\Lambda_t := t\mathbb{Z} + t^{-1}\mathrm{i}\mathbb{Z}$, a self-dual lattice, with the semicharacter $\alpha_t(ta + t^{-1}b\mathrm{i}) := (-1)^{ab}$.

Problem 12.2.4. $\mathcal{T}(\Lambda_t, \alpha_t) = \mathbb{C}\,e^{\pi z^2/2}\,\vartheta_{\mathrm{i}/t^2}(z/t)$.

For every self-dual Λ and every semicharacter α of it, $\mathcal{T}(\Lambda, \alpha)$ is a line. More generally, for a positive definite Hermitian form in g variables whose imaginary part E is integral on a full lattice Λ, the dimension is the absolute value of the Pfaffian of the matrix of E in a lattice basis [Pol, Ch. 3]. Section 12.3 deduces the line statement from Problem 12.2.1 for pairs compatible with $(\mathbb{Z}[\mathrm{i}], \alpha_0)$.

12.3. Intertwiners

Two pairs (Λ_1, α_1) and (Λ_2, α_2) as in Section 12.1 are *compatible* if Λ_1 and Λ_2 are commensurable and $\alpha_1 = \alpha_2$ on $\Lambda_{12} := \Lambda_1 \cap \Lambda_2$. Since both lattices have covolume 1, $n_{12} := [\Lambda_2 : \Lambda_{12}]$ equals $[\Lambda_1 : \Lambda_{12}]$. The *intertwiner* is

$$I_{12}\,G := n_{12}^{-1/2} \sum_{y \in \Lambda_2/\Lambda_{12}} \alpha_2(y)\,U_y G \qquad (G \in \mathcal{T}(\Lambda_1, \alpha_1)).$$

It is a finite sum of translation operators. The theory goes back to Weil [We]; see [Pol, Ch. 4] and [LV].

Problem 12.3.1. Each summand of $I_{12}G$ depends only on the class of y modulo Λ_{12}, and I_{12} maps $\mathcal{T}(\Lambda_1, \alpha_1)$ into $\mathcal{T}(\Lambda_2, \alpha_2)$.

Problem 12.3.2. $I_{21}I_{12}$ is the identity of $\mathcal{T}(\Lambda_1,\alpha_1)$.

Problem 12.3.3. $\|I_{12}G\|_{\Lambda_2} = \|G\|_{\Lambda_1}$.

By Problems 12.2.1 and 12.3.2, $\mathcal{T}(\Lambda,\alpha)$ is a line, spanned by IG for the intertwiner I from $\mathbb{Z}[\mathrm{i}]$, whenever (Λ,α) is compatible with $(\mathbb{Z}[\mathrm{i}],\alpha_0)$.

Problem 12.3.4. Let p be a prime, and let I_{12} be the intertwiner from $(\mathbb{Z}[\mathrm{i}],\alpha_0)$ to $(\Lambda_{1/p},\alpha_{1/p})$. Then

$$I_{12}\,G = \sqrt{p}\;e^{\pi z^2/2}\,\vartheta_{p^2\mathrm{i}}(pz),$$

and Problem 12.3.2 for this pair is the splitting of $\vartheta_{\mathrm{i}}(z)$ according to the class of n modulo p.

12.4. Composition

Is $I_{23}I_{12}$ equal to I_{13}? Let (Λ_i,α_i), $i=1,2,3$, be pairwise compatible pairs, each compatible with $(\mathbb{Z}[\mathrm{i}],\alpha_0)$. For $z\in\Lambda_3$ and $y\in\Lambda_2$ with $z+y\in\Lambda_1$ put

$$\psi(z,y) := \alpha_3(z)\,\alpha_2(y)\,\alpha_1(z+y)\,e^{-\pi\mathrm{i}E(z,y)}.$$

Problem 12.4.1. Let S_2 and S_3 be sets of representatives of Λ_2/Λ_{12} and Λ_3/Λ_{23}. There is a unique scalar $\mu=\mu(\Lambda_1,\Lambda_2,\Lambda_3)$, with the three semicharacters understood, for which

$$I_{23}I_{12} = \mu\, I_{13}, \qquad \mu = \Big(\frac{n_{13}}{n_{12}\,n_{23}}\Big)^{1/2} \sum_{\substack{z\in S_3,\ y\in S_2\\ z+y\in\Lambda_1}} \psi(z,y).$$

Problem 12.4.2. $|\mu|=1$, $\mu(\Lambda_1,\Lambda_2,\Lambda_1)=1$, and $\mu(\Lambda_1,\Lambda_3,\Lambda_2)=\overline{\mu(\Lambda_1,\Lambda_2,\Lambda_3)}$.

Problem 12.4.3. For four pairwise compatible pairs, each compatible with $(\mathbb{Z}[\mathrm{i}],\alpha_0)$,

$$\mu(\Lambda_1,\Lambda_2,\Lambda_3)\,\mu(\Lambda_1,\Lambda_3,\Lambda_4) = \mu(\Lambda_2,\Lambda_3,\Lambda_4)\,\mu(\Lambda_1,\Lambda_2,\Lambda_4).$$

Problems 12.3.2, 12.3.3, 12.4.1 and 12.4.2 prove the following. Weil computes the factor system for his projective representation [We]. The formula for μ appears in [Adr, Lemma 7.3(ii)] with a single semicharacter on all three lattices.

Theorem 12.4.1. *The intertwiners are isometries, $I_{21}=I_{12}^{-1}$, and $I_{23}I_{12}=\mu\,I_{13}$ with $|\mu|=1$ given by the sum of Problem 12.4.1.*

The constant μ is a normalized quadratic Gauss sum. For the Weil representation, Lion and Vergne express the analogous constant through the Maslov index of three Lagrangian subspaces [LV]; Gauss sums of integral quadratic forms are treated in [Pol, App. B]. The next section computes μ in the smallest cases.

12.5. Neighbours of index p and 2

Odd index. Let $R\subset\mathbb{Q}$ be the ring of fractions with odd denominator, so that $R/2R=\mathbb{Z}/2$, and for $r\in R$ let $(-1)^r := (-1)^n$ for any integer $n\equiv r \pmod{2R}$. For $v=x+\mathrm{i}y$ with $x,y\in R$ put $\alpha_R(v):=(-1)^{xy}$.

Problem 12.5.1. α_R is a semicharacter of every self-dual lattice contained in $R+R\mathrm{i}$. It equals α_0 on $\mathbb{Z}[\mathrm{i}]$ and $\alpha_{1/p}$ on $\Lambda_{1/p}$ for p odd.

So all self-dual lattices in $R + R\mathrm{i}$, with α_R, are pairwise compatible. Fix an odd prime p, and let $N := \mathbb{Z} + p\mathbb{Z}\mathrm{i}$.

Problem 12.5.2. The self-dual lattices Λ with $N \subset \Lambda$ are $\Lambda^{(\infty)} := \mathbb{Z}[\mathrm{i}]$ and $\Lambda^{(c)} := N + \mathbb{Z}(\frac{1}{p} + c\mathrm{i})$ for $c \in \mathbb{Z}/p$. Moreover $\Lambda^{(0)} = \Lambda_{1/p}$, and any two of these $p+1$ lattices intersect in N.

Problem 12.5.3. For $b \neq c$ in $\mathbb{Z}/p$, with $\bar{2} := (p+1)/2$,

$$\mu(\Lambda^{(\infty)}, \Lambda^{(b)}, \Lambda^{(c)}) = \frac{1}{\sqrt{p}} \sum_{t \bmod p} e^{2\pi\mathrm{i}\,\bar{2}(c-b)t^2/p} = \Big(\frac{2(c-b)}{p}\Big)\,\varepsilon_p.$$

For $p = 3$, $b = 0$, $c = 1$ this is $-\mathrm{i}$.

Problem 12.5.4. For distinct $a, b, c \in \mathbb{Z}/p$,

$$\mu(\Lambda^{(a)}, \Lambda^{(b)}, \Lambda^{(c)}) = \Big(\frac{2(b-a)(c-b)(c-a)}{p}\Big)\,\varepsilon_p.$$

Index 2. Here α_R is not available, and the semicharacters must be chosen. Let $N_2 := \mathbb{Z} + 2\mathbb{Z}\mathrm{i}$, let $\Lambda' := \Lambda_{1/2} = \frac{1}{2}\mathbb{Z} + 2\mathbb{Z}\mathrm{i}$ with $\alpha' := \alpha_{1/2}$, and let $\Lambda'' := \mathbb{Z}(\frac{1}{2} + \mathrm{i}) + \mathbb{Z}$ with $\alpha''(a(\frac{1}{2} + \mathrm{i}) + m) := (-1)^{am}$.

Problem 12.5.5. $\mathbb{Z}[\mathrm{i}]$, Λ' and Λ'' are the self-dual lattices containing N_2, and any two of them intersect in N_2. The pairs $(\mathbb{Z}[\mathrm{i}], \alpha_0)$, (Λ', α') and (Λ'', α'') are pairwise compatible, and each of Λ', Λ'' has exactly two semicharacters compatible with α_0.

Problem 12.5.6. $\mu(\mathbb{Z}[\mathrm{i}], \Lambda', \Lambda'') = 2^{-1/2} \sum_{t \bmod 2} e^{\pi\mathrm{i}t^2/2} = e^{\pi\mathrm{i}/4}$. If α'' is replaced by the other semicharacter compatible with α_0, then $\mu = e^{-\pi\mathrm{i}/4}$.

For the distinct neighbours above, the composition scalar is a fourth root of unity at an odd prime and a primitive eighth root of unity at index 2. In both calculations the phase $\psi(z, y)$ contains the semicharacter values and the factor $e^{-\pi\mathrm{i}E(z,y)}$. At index 2 the choice of compatible semicharacter distinguishes the two eighth roots in Problem 12.5.6.

Unitary changes of variable. Constants of the same kind appear when a lattice is moved by a unitary map ϱ of V. Let $G' \in \mathcal{T}(\Lambda, \alpha)$. If ϱ preserves Λ and α, then $G' \circ \varrho^{-1}$ lies in the same line. For $z \mapsto \mathrm{i}z$ on $(\mathbb{Z}[\mathrm{i}], \alpha_0)$ the scalar is 1 by Problem 12.2.2. If the transported pair $(\varrho\Lambda, \alpha \circ \varrho^{-1})$ is compatible with (Λ, α), then $G' \circ \varrho^{-1}$ and $I_{\Lambda \to \varrho\Lambda}\, G'$ belong to the same theta line and have the same norm. For $G' \neq 0$ they differ by a scalar of absolute value one. Related composition factors occur in Weil's construction of the metaplectic cover [We]. For a completion $F \neq \mathbb{C}$ of a number field containing the n-th roots of unity, Kubota gives a factor set for an n-fold cover of $\mathrm{SL}_2(F)$ in terms of Hilbert symbols [Kub].

In the Lean formalization. In the Lean proof [GL], the kernel of the definite lift from X_j to X_{j+1} for a tame character (`liftKernel`) takes data in the finite part (`finitePart`) for the Hermitian space of maps from X_j to X_{j+1}. This is the complex span of the finite Heisenberg translates of that space's vacuum. A translation tuple x (`Translations`) consists, for every $j \in \mathbb{N}$, of a matrix x_j over the finite adeles of the CM field, with rows indexed by the basis of X_{j+1} and columns by the basis of X_j. The value `translateDatum` $x\, j$ is the translate by x_j of the integral vacuum of that Hermitian space (Chapter 34).

The corresponding statements of the Lean proof for the data of the composite lift are `gluedDoubleLiftDatum`, `doubleLiftGluing`, `gluedVacuum` and `gluedTensor`, and for the composite lift itself `compositeLift` and `restrictToY`.

These declarations are in `grh/src/New/UpperBound/Objects/Columns.lean`, with `finitePart` in `grh/src/Lib/Weil/Construction/VacuumProjections.lean`.

Hints

§12.1.

12.1.1: $|(U_xG)(z)|^2e^{-\pi|z|^2} = |G(z+x)|^2e^{-\pi|z+x|^2}$. For the product, compare exponents; $H(x,y) - \operatorname{Re} H(x,y) = \mathrm{i}E(x,y)$.

12.1.2: Polar coordinates.

12.1.3: Two semicharacters differ by a character of Λ with values ±1. For α_0 check the identity on the basis $1, \mathrm{i}$ and extend.

12.1.4: The first statement is the definition of a semicharacter together with Problem 12.1.1. For the second, commute U_λ past U_w.

12.1.5: Compare the Gaussian integrals over translates of one fundamental parallelogram.

§12.2.

12.2.1: For $F \in \mathcal{T}(\mathbb{Z}[\mathrm{i}], \alpha_0)$, $e^{-\pi z^2/2}F$ has period 1; expand it in a Fourier series and use the law under $z \mapsto z + \mathrm{i}$ to relate consecutive coefficients.

12.2.2: Apply Problem 12.2.1 to $G(\mathrm{i}z)$.

12.2.3: Integrate first over $x \in [0,1]$; the remaining sum over n unfolds $[0,1]$ in y to $\mathbb{R}$.

12.2.4: As in Problem 12.2.1, after the substitution $z = tw$.

§12.3.

12.3.1: Use the homomorphism in Problem 12.1.4 for (Λ_2, α_2). First compare the summands at y and $y+\lambda$ for $\lambda \in \Lambda_{12}$. Then apply $\alpha_2(w)U_w$ for $w \in \Lambda_2$ and inspect its effect on the cosets.

12.3.2: By Problem 12.1.4, $\sum_{x \in \Lambda_1/\Lambda_{12}} \alpha_1(x)U_x$ multiplies U_yG by a character sum. The character $x \mapsto e^{-2\pi\mathrm{i}E(x,y)}$ of Λ_1/Λ_{12} is trivial exactly when $y \in \Lambda_1$, since Λ_1 is self-dual.

12.3.3: Integrate over $\mathbb{C}/\Lambda_{12}$, an n_{12}-fold cover of both tori. A cross term $U_yG\,\overline{U_{y'}G}\,e^{-\pi|z|^2}$ with $y \not\equiv y'$ is Λ_{12}-periodic and changes by a nontrivial character under Λ_1.

12.3.4: Representatives s/p, $0 \le s < p$. Then $\sum_s e^{2\pi\mathrm{i}ns/p}$ picks out $p \mid n$.

§12.4.

12.4.1: Use Problems 12.2.1 and 12.3.2 to compare the two maps between the source and target theta spaces. Apply $\sum_x \alpha_1(x)U_x$ to both sides of $I_{23}I_{12}G = \mu\, I_{13}G$, with x running over $\Lambda_1/(\Lambda_1 \cap \Lambda_2 \cap \Lambda_3)$, and use Problem 12.1.4.

12.4.2: For the modulus use Problem 12.3.3. Then compute $I_{32}I_{23}I_{12}$ in two ways.

12.4.3: Compute $I_{34}I_{23}I_{12}$ in two ways.

§12.5.

12.5.1: $(x+x')(y+y') - xy - x'y' - E(v,v') = 2xy'$.

12.5.2: The dual of N for E is $N^\perp = \frac{1}{p}\mathbb{Z} + \mathbb{Z}\mathrm{i}$, and $N^\perp/N \cong (\mathbb{Z}/p)^2$. A self-dual $\Lambda \supset N$ lies in $N^\perp$ and corresponds to a line in $N^\perp/N$.

12.5.3: Take $z = t(\frac{1}{p} + c\mathrm{i})$ and $y = -t(\frac{1}{p} + b\mathrm{i})$. Then $z + y \in \mathbb{Z}[\mathrm{i}]$, and $\psi(z,y)$ depends only on the classes of z and y modulo N.

12.5.4: Problem 12.4.3 with $\Lambda_1 = \mathbb{Z}[\mathrm{i}]$, and Problem 12.5.3.

12.5.5: As in Problem 12.5.2 with $p = 2$. Note that $\alpha_0 = 1$ on N_2, and apply Problem 12.1.3's argument to Λ' and Λ''.

12.5.6: One term has $z = y = 0$. In the other, z represents the nonzero class of Λ''/N_2.

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