THE PLACES, THE WEIGHTS AND THE REMAINDER

Prerequisites. From Chapter 1, *From zeros to a one-sided bound*: the weight λ, the constant $c_{d,q}$ and the remainder $R_{d,q}$ of equation (4), recalled below, and Problem 1.4.7. From Chapter 2, *Inert primes and the choice of d*: Section 2.1, on the values of $1-\chi_K$. Cyclotomic fields and their Galois groups [Wa, Ch. 2]; infinite places of a number field as conjugation orbits of embeddings into $\mathbb{C}$ [Ne, Ch. II, §8]; quadratic reciprocity [SeA, Ch. I, §3].

In the proof, the main inequality (Theorem 35.2.1) bounds from above a centered sum over the infinite places of class 1 modulo q_0 in the CM field obtained by adjoining a root of unity to $K=\mathbb{Q}(\sqrt{-d})$. Chapter 1 needs an upper bound for the remainder of a weighted sum over the prime powers $n\equiv 1 \pmod{q_0}$ up to x. How does the sum over places give such a bound? An exact finite-sum identity gives the answer. In the summand indexed by n, the weight at the translated place sets the cutoff. For an admissible datum the centered total over places of class 1 modulo q_0 equals a positive multiple of $R_{q_0}(x)$ plus two terms at x_1 (Theorem 3.2.1).

Throughout, $d\equiv 3 \pmod 4$ is a prime, $K:=\mathbb{Q}(\sqrt{-d})$, and $\chi_K(n):=\left(\frac{n}{d}\right)$ is the Legendre symbol (the χ_d of Chapter 1), so $\chi_K(d)=0$. For an integer $q\geq 1$ prime to d we write, as in Chapter 1,

$$\lambda_q(n):=\begin{cases}\Lambda(n)\big(1-\chi_K(n)\big), & n\equiv 1 \pmod q,\\ 0, & \text{otherwise,}\end{cases}\qquad R_q(z):=\sum_{1\leq n\leq z}\lambda_q(n)\Big(\frac{1}{n}-\frac{1}{z}\Big)-\frac{\log z}{\varphi(q)}-c_q$$

for real $z>0$, where $c_q:=c_{d,q}$ and $R_q:=R_{d,q}$ are the constant and the remainder of equation (4), and λ_q is the weight λ there for the modulus q; we drop d, which is fixed. The value of c_q plays no role in the identities of this chapter. We write $u_+:=\max(u,0)$ for real u.

3.1. The host field and its places

Let $q_0\geq 3$ and let q_1 be a multiple of q_0 such that every prime factor of q_1 divides q_0, and $d\nmid q_1$. Let $E:=K(\zeta)$ with ζ a primitive q_1-th root of unity, regarded as a subfield of $\mathbb{C}$. For $g\in \mathrm{Gal}(E/K)$ let $e(g)\in(\mathbb{Z}/q_1\mathbb{Z})^\times$ be the class with $g(\zeta)=\zeta^{e(g)}$.

Problem 3.1.1. The map $e\colon \mathrm{Gal}(E/K)\to(\mathbb{Z}/q_1\mathbb{Z})^\times$ is an isomorphism, and $[E:\mathbb{Q}]=2\varphi(q_1)$.

Let Φ be the set of infinite places of E and $f:=|\Phi|$. For $g\in\mathrm{Gal}(E/K)$ let $\langle g\rangle\in\Phi$ be the place of the embedding $g\colon E\to\mathbb{C}$.

Problem 3.1.2. Every infinite place of E is complex, and $g\mapsto\langle g\rangle$ is a bijection $\mathrm{Gal}(E/K)\to\Phi$. Hence $f=\varphi(q_1)$.

We make Φ a group through $g\mapsto\langle g\rangle$, and identify it with $(\mathbb{Z}/q_1\mathbb{Z})^\times$ through e: the place $\sigma\in\Phi$ has *label* $v\in(\mathbb{Z}/q_1\mathbb{Z})^\times$ if $\sigma=\langle g\rangle$ with $e(g)=v$. The place with label 1 is the identity. For $\nu\in\mathbb{Z}$ let $[\nu]\in\Phi$ be the place with label $\nu \bmod q_1$

Date: November 26, 2026.

if $\gcd(\nu, q_1) = 1$, and the identity place otherwise. Let $\mathrm{cl}\colon \Phi \to (\mathbb{Z}/q_0\mathbb{Z})^\times$ be the label reduced modulo q_0, and let

$$\Phi_a := \mathrm{cl}^{-1}(1),$$

the *a-places*. Fix a set $\Phi_b \subseteq \mathrm{cl}^{-1}(-1)$ of *b-places*. We write $\#$ for the number of elements.

Problem 3.1.3. $\#\Phi_a = q_1/q_0$ and $f = \#\Phi_a\, \varphi(q_0)$.

The place attached to an integer $n \geq 1$ and an a-place τ is $[-n]\tau$: for n prime to q_1, its label is $-n$ times the label of τ.

Problem 3.1.4. For $\tau \in \Phi_a$ and every integer $n \geq 1$, $\mathrm{cl}\big([-n]\tau\big) = -1$ if and only if $n \equiv 1 \pmod{q_0}$.

Problem 3.1.5. For every integer $n \geq 1$, the number of $\tau \in \Phi_a$ with $[-n]\tau \in \Phi_b$ is $\#\Phi_b$ if $n \equiv 1 \pmod{q_0}$, and 0 otherwise.

When $\gcd(n, q_1) > 1$, our convention gives $[-n]\tau = \tau$. Under the hypotheses of the next problem such n have coefficient $\lambda_1(n) = 0$ in the prime sums below, so the convention does not affect them.

Problem 3.1.6. (i) If $16 \mid q_0$, $q_0 \mid 2(d+1)$ and $q_0 \nmid d+1$, then $d \equiv 7 \pmod 8$ and $\chi_K(p) = 1$ for every prime $p \mid q_0$. (ii) In that case $\lambda_1(n) = 0$ for every $n \geq 1$ with $\gcd(n, q_1) > 1$.

The *toy example* is $q_0 = 4$, $q_1 = 16$, $d = 7$, so that $f = 8$, Φ_a consists of the places with labels $1, 5, 9, 13$, and Φ_b consists of the place with label 15, so $\#\Phi_a = 4$ and $\#\Phi_b = 1$. It does not satisfy the hypotheses of Problem 3.1.6(i), but $\chi_K(2) = (2/7) = 1$, and the conclusion of Problem 3.1.6(ii) holds.

Problem 3.1.7. In the toy example, the primes $p \leq 61$ with $\lambda_1(p) \neq 0$ are 7 and the primes inert in K, and the label of $[-p]$ is as follows.

label	1	3	5	7	9	11	13	15
p	31, 47	13, 61	59	41	7	5	3, 19	17

The other $n \leq 61$ with $\lambda_1(n) \neq 0$ are 27 and 49, with $\lambda_1(27) = 2\log 3$ and $\lambda_1(49) = \log 7$.

3.2. Two weights and the prime sum

Let $x < x_1$ and b be positive integers with $x_1 \mid b$, and let $a := xb$. The *weight* of a place is

$$\kappa(\sigma) := \begin{cases} a/x = b, & \sigma \in \Phi_b, \\ a/x_1, & \sigma \notin \Phi_b, \end{cases}$$

an integer; the b-places carry the larger weight. The *prime sum* at a place σ is, for real y,

$$s_{\mathrm{fin}}(y, \sigma) := \sum_{n=1}^{a} \lambda_1(n) \Big(\frac{y}{n} - \kappa\big([-n]\sigma\big) \Big)_+ .$$

For real $z > 0$ and $n \geq 1$ let $H_z(n) := (a/n - a/z)_+$.

Problem 3.2.1. $H_z(n) = a(1/n - 1/z)$ for $n < z$, and $H_z(n) = 0$ for $n \geq z$. Hence for $1 \leq n \leq a$ the n-th term of $s_{\mathrm{fin}}(a, \sigma)$ is $\lambda_1(n) H_x(n)$ if $[-n]\sigma \in \Phi_b$, and $\lambda_1(n) H_{x_1}(n)$ otherwise.

Problem 3.2.2. Let $\sigma \in \Phi$. (i) For $0 \le y \le a$ and $x_1 \le n \le a$, the n-th term of $s_{\mathrm{fin}}(y,\sigma)$ is 0. (ii) For $x \le n < x_1$, the n-th term of $s_{\mathrm{fin}}(a,\sigma)$ is 0 if $[-n]\sigma \in \Phi_b$, and $\lambda_1(n)\, a(1/n - 1/x_1)$ otherwise, which is positive unless $\lambda_1(n) = 0$.

The first step towards the total over the a-places is a ramp sum in a progression.

Problem 3.2.3. For integers $q \ge 1$ prime to d, $A_0 \ge 1$, and real $0 < z \le A_0$,

$$\sum_{n=1}^{A_0} \lambda_q(n)\Big(\frac{A_0}{n} - \frac{A_0}{z}\Big)_+ = A_0\Big(R_q(z) + \frac{\log z}{\varphi(q)} + c_q\Big).$$

For $d = 7$, $q = 4$, $A_0 = 60$, $z = 20$, both sides equal $18\log 5 + \frac{42}{13}\log 13 + \frac{18}{17}\log 17 = 40.2565\ldots$.

Problem 3.2.4. For every integer $n \ge 1$,

$$\sum_{\tau \in \Phi_a} \lambda_1(n)\Big(\frac{a}{n} - \kappa\big([-n]\tau\big)\Big)_+ = \#\Phi_a\, \lambda_1(n) H_{x_1}(n) + \#\Phi_b\, \lambda_{q_0}(n)\big(H_x(n) - H_{x_1}(n)\big).$$

Problem 3.2.5.

$$\sum_{\tau \in \Phi_a} s_{\mathrm{fin}}(a,\tau) = a\Big[\#\Phi_b\big(R_{q_0}(x) - R_{q_0}(x_1)\big) + \#\Phi_a R_1(x_1)\Big] + a\Big[\#\Phi_a(\log x_1 + c_1) + \#\Phi_b\,\frac{\log(x/x_1)}{\varphi(q_0)}\Big].$$

Problem 3.2.6. In the toy example with $x = 20$, $x_1 = 60$, $b = 960$, so that $a = 19200$ and the weights are 960 and 320, the four prime sums $s_{\mathrm{fin}}(a,\tau)$ at the places τ with labels $1, 5, 9, 13$ rounded to the nearest hundredth, are 45108.01, 45591.18, 47772.92, 46814.23. Their total and the right side of Problem 3.2.5 both round to 185286.35.

The expected value uses the mean of the logarithms of the weights over all places. We define the *expected value*

$$s_{\mathrm{main}} := a\Big(\log a - \frac{1}{f}\sum_{\sigma \in \Phi} \log \kappa(\sigma) + c_1\Big).$$

Problem 3.2.7. $\sum_{\sigma\in\Phi} \log\kappa(\sigma) = f\log(a/x_1) - \#\Phi_b \log(x/x_1)$. In the toy example both sides equal $47.24518\ldots$.

Problem 3.2.8. $\#\Phi_a\, s_{\mathrm{main}} = a\big[\#\Phi_a(\log x_1 + c_1) + \#\Phi_b \log(x/x_1)/\varphi(q_0)\big]$.

The *total remainder* is $\mathrm{rem} := \sum_{\tau\in\Phi_a}\big(s_{\mathrm{fin}}(a,\tau) - s_{\mathrm{main}}\big)$, so that the mean of $s_{\mathrm{fin}}(a,\tau)$ over Φ_a is $s_{\mathrm{main}} + \mathrm{rem}/\#\Phi_a$. The total remainder satisfies

$$\mathrm{rem} = a\Big[\#\Phi_b\big(R_{q_0}(x) - R_{q_0}(x_1)\big) + \#\Phi_a R_1(x_1)\Big] \tag{1}$$

in the setting of this chapter. In the toy example, $\mathrm{rem} + 76800\, c_1 = -118612.63\ldots$.

In the proof the *parameters* (q_0, d) are a positive multiple q_0 of 18480 and a prime d with $q_0 \mid 2(d+1)$ and $q_0 \nmid d+1$ (Definition 33.1.1 of Chapter 33). An *admissible datum* for them is a datum satisfying all the conditions of Definition 33.3.1 there. We record the components used here. The scales x, x_1, b are positive integers. The number m of theta lifts in each chain is a power of 2 with $\sqrt{x} \le m \le 2\sqrt{x}$, and the tower exponent is $s = 22\log_2 m$. With $e_s := \lfloor \log_3 s \rfloor$, let $q_1 := q_0\, 3^{e_s+5} 2^s$; the field E, its embedding into $\mathbb{C}$, the primitive root ζ and the labels of the places are as in §3.1. We have $q_1 x_1 \mid b$. The datum supplies $\Phi_b \subseteq \mathrm{cl}^{-1}(-1)$ with $q_1/q_0 \le \#\Phi_b \log b$, and the weight κ above.

Theorem 3.2.1. *For all parameters (q_0, d) and every admissible datum for them,* (1) *holds, where $a = xb$ and R_q and c_1 are those of Chapter 1 for the prime d.*

For these data q_0 and q_1 have the same prime factors, $d \nmid q_1$, and $d \equiv 7 \pmod 8$. The scale conditions give $0 < \log x$ and $10^6 \log x \le \log x_1$, so $x < x_1$.

3.3. The remainder at the first scale

Chapter 1 needs an upper bound for $R_{q_0}(x)$. When Φ_b is nonempty, equation (1) isolates $R_{q_0}(x)$ at the price of two terms at x_1.

Problem 3.3.1. If $\#\Phi_b > 0$, then $R_{q_0}(x) = \dfrac{\mathrm{rem}}{a\,\#\Phi_b} + R_{q_0}(x_1) - \dfrac{\#\Phi_a}{\#\Phi_b} R_1(x_1)$.

For $\#\Phi_b > 0$, the coefficient of $R_1(x_1)$ has magnitude $\#\Phi_a/\#\Phi_b$, the reciprocal of the proportion of the -1 fibre occupied by Φ_b; this is why Φ_b must be dense.

Problem 3.3.2. If $b > 1$ and $\#\Phi_a \le \#\Phi_b \log b$, then $\#\Phi_b > 0$ and

$$R_{q_0}(x) \le \frac{\mathrm{rem}}{a\,\#\Phi_b} + |R_{q_0}(x_1)| + \log b\,|R_1(x_1)|.$$

In the toy example $\#\Phi_a/\#\Phi_b = 4 \le \log 960 = 6.8669\ldots$.

Problem 3.3.3. Suppose $b > 1$, $\#\Phi_a \le \#\Phi_b \log b$, and m is a positive integer. If $m \cdot \mathrm{rem} \le fa(\log b)^5$, then

$$R_{q_0}(x) \le \frac{\varphi(q_0)(\log b)^6}{m} + |R_{q_0}(x_1)| + \log b\,|R_1(x_1)|.$$

The hypothesis $m \cdot \mathrm{rem} \le fa(\log b)^5$ is the *main inequality*. Chapter 35 proves it for every admissible datum (Theorem 35.2.1) and derives the bound of Problem 3.3.3 in Problem 35.3.1, before passing to the asymptotic estimate of Theorem 35.3.1; Chapter 4 constructs a set Φ_b dense enough for Problem 3.3.2.
In the Lean formalization. In the Lean proof [GL], the parameters and the admissible data are `New.Parameters` and `New.Data`; Theorem 3.2.1 is `New.Data.totalRemainder_eq`, in `New/Reduction/PrimeSums.lean`; and the conclusion of Problem 3.3.3 for every admissible datum, without further hypotheses, is `New.Data.primeRemainder_le`, in `New/Reduction/OneSidedBound.lean`.

Hints

§3.1.

- **3.1.1:** d ramifies in K and not in $\mathbb{Q}(\zeta)$, so $K \cap \mathbb{Q}(\zeta) = \mathbb{Q}$ [Wa, Ch. 2].
- **3.1.2:** Complex conjugation restricts to the nontrivial automorphism of K.
- **3.1.3:** cl is a surjective homomorphism, and $\varphi(q_1)/\varphi(q_0) = q_1/q_0$ when q_0 and q_1 have the same prime factors.
- **3.1.4:** cl is a homomorphism; if $\gcd(n, q_1) > 1$, then $n \not\equiv 1 \pmod{q_0}$.
- **3.1.5:** Multiplication by $[-n]$ is a bijection of Φ; Problem 3.1.4.
- **3.1.6:** (i) $d + 1 \equiv q_0/2 \pmod{q_0}$; for each odd prime $p \mid q_0$, $d \equiv -1 \pmod p$, and reciprocity. (ii) Section 2.1.
- **3.1.7:** p is inert in $\mathbb{Q}(\sqrt{-7})$ exactly when $p \equiv 3, 5, 6 \pmod 7$.

§3.2.

3.2.2: (i) Compare y/n and $\kappa([-n]\sigma)$ with a/x_1. (ii) Use the two possible weights at $[-n]\sigma$.

3.2.3: Problem 3.2.1 with A_0 in place of a, and the definition of R_q.

3.2.4: Write $(a/n - \kappa(\sigma))_+$ as $H_{x_1}(n)$ plus a correction supported on Φ_b; Problem 3.1.5, and $\lambda_{q_0}(n) = \lambda_1(n)$ for $n \equiv 1 \pmod{q_0}$.

3.2.5: Exchange the sums over τ and n; Problem 3.2.3 three times, with $(q, z) = (1, x_1), (q_0, x), (q_0, x_1)$.

3.2.6: Only the $n < 60$ with $\lambda_1(n) \neq 0$ contribute (Problems 3.1.7 and 3.2.2).

3.2.8: Problems 3.1.3 and 3.2.7.

§3.3.

3.3.1: Equation (1).

3.3.2: $\#\Phi_a > 0$.

3.3.3: Problem 3.1.3: $f/\#\Phi_b = \varphi(q_0)\,\#\Phi_a/\#\Phi_b$.

Theorems.

3.2.1: Compare Problems 3.2.5 and 3.2.8.

References

[Ne] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. **322**, Springer, 1999.

[SeA] J.-P. Serre, *A Course in Arithmetic*, Grad. Texts in Math. **7**, Springer, 1973.

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. **83**, Springer, 1997.