DEFINITE THETA LIFTS AND THEIR NORMS

Prerequisites. Chapter 21, *Masses and the count of forms*: the class sets Cl_n, the groups Γ_L, the representations $V_\lambda(L) = \mathbb{S}_\lambda(L_\mathbb{C})$, the notation e^ν in Sym^a, and the spaces of forms $\mathcal{A}_\lambda(n)$ with their norm (Sections 21.1 and 21.3). From Chapter 11: Theorem 11.3.1 and Problems 11.3.7, 11.3.1 and 11.3.4. From Chapter 15: Section 15.1 and Problem 15.1.3. Stirling's formula.

Since a definite unitary group has a finite class set, a theta lift between two such groups is a finite matrix. What are its entries, and how large is it? We compute both in the smallest cases.

23.1. The theta kernel

How does one map forms on one class set to forms on another? The idea is to sum over all homomorphisms between the two lattices, with a Gaussian weight.

Let L and L' be positive definite unimodular Hermitian lattices over $\mathcal{O} = \mathbb{Z}[\mathrm{i}]$ of ranks n and n'. For $x \in \mathrm{Hom}_\mathcal{O}(L, L')$ let $x^* \in \mathrm{Hom}(L'_\mathbb{C}, L_\mathbb{C})$ be its adjoint, and let $h(x, y) := \mathrm{tr}(y^* x)$. Let

$$\Theta_{L,L'}(Z) := \sum_{x \in \mathrm{Hom}_\mathcal{O}(L,L')} e^{-\pi h(x,x) + \mathrm{tr}(Zx)}, \qquad Z \in \mathrm{Hom}(L'_\mathbb{C}, L_\mathbb{C}).$$

For a polynomial P in the entries of x, let $P(\partial)$ be the differential operator obtained by replacing $x_{\beta\alpha}$ with $\partial/\partial Z_{\alpha\beta}$. For a weight λ define

$$\vartheta_\lambda(L, L') := \sum_{x \in \mathrm{Hom}_\mathcal{O}(L,L')} \mathbb{S}_\lambda(x)\, e^{-\pi h(x,x)} \colon V_\lambda(L) \to V_\lambda(L'),$$

and, for the class sets Cl_n and $\mathrm{Cl}_{n'}$, the *lift* $\Theta_\lambda \colon \mathcal{A}_\lambda(n) \to \mathcal{A}_\lambda(n')$,

$$(\Theta_\lambda \varphi)(L') := \sum_{L \in \mathrm{Cl}_n} \frac{1}{|\Gamma_L|}\, \vartheta_\lambda(L, L')\, \varphi(L). \tag{1}$$

In bases, Θ_λ is a finite matrix whose block in row L' and column L is $|\Gamma_L|^{-1}\vartheta_\lambda(L, L')$, restricted from $V_\lambda(L)^{\Gamma_L}$ to $V_\lambda(L')^{\Gamma_{L'}}$. For a matrix coefficient P of $\mathbb{S}_\lambda$, the series $\sum_x P(x)\, q^{h(x,x)}$ is a classical theta series with polynomial coefficients, and the corresponding entry of ϑ_λ is its value at $q = e^{-\pi}$.

Thus $\Theta_{L,L'}$ is a theta function of the Hermitian lattice $\mathrm{Hom}_\mathcal{O}(L, L')$, which is unimodular of rank nn' by Problem 23.1.1 below. Chapter 12, *The Fock model and theta intertwiners*, treats one-variable theta functions of self-dual lattices with semicharacters; for commensurable lattices whose semicharacters agree on their intersection it defines an intertwiner, and for the neighbours of $\mathbb{Z}[\mathrm{i}]$ studied there the composition scalar is a root of unity given by a Gauss sum. The problems of this chapter use only the series itself.

Problem 23.1.1. If L and L' are unimodular, then with $h(x, y) = \mathrm{tr}(y^* x)$, $\mathrm{Hom}_\mathcal{O}(L, L')$ is a unimodular Hermitian lattice of rank nn', and $\mathrm{Hom}_\mathcal{O}(I_n, I_{n'}) \cong I_{nn'}$.

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Problem 23.1.2. $\Theta_{L,L'}$ is entire, and $P(\partial)\Theta_{L,L'}(0) = \sum_x P(x)\, e^{-\pi h(x,x)}$. If $\Theta_{L,L'}(Z) = \sum_\nu b_\nu Z^\nu$ and $P(x) = \sum_\nu P_\nu x^\nu$, with the matching indexing of the entries of Z and of x, this equals $\sum_\nu P_\nu\, \nu!\, b_\nu$.

Problem 23.1.3. In Schur bases of $V_\lambda(L)$ and $V_\lambda(L')$ obtained over $\mathbb{Q}$ from $\mathcal{O}$-bases of L and L', each entry of $\vartheta_\lambda(L,L')$ is $P(\partial)\Theta_{L,L'}(0)$ for a polynomial P, homogeneous of degree $|\lambda|$, with rational coefficients. Moreover $\vartheta_\lambda(L,L') = 0$ unless $4 \mid |\lambda|$.

Problem 23.1.4. For $\gamma \in \Gamma_L$ and $\gamma' \in \Gamma_{L'}$, $\mathbb{S}_\lambda(\gamma')\,\vartheta_\lambda(L,L') = \vartheta_\lambda(L,L') = \vartheta_\lambda(L,L')\,\mathbb{S}_\lambda(\gamma)$. Hence (1) is a form on $\mathrm{Cl}_{n'}$, and it does not depend on the lattices chosen in the classes, up to transport of forms along isometries.

Problem 23.1.5. $\vartheta_\lambda(L,L')^* = \vartheta_\lambda(L',L)$. Hence the lift $\Theta'_\lambda\colon \mathcal{A}_\lambda(n') \to \mathcal{A}_\lambda(n)$ in the other direction is the adjoint of Θ_λ: $\langle \Theta_\lambda\varphi, \psi\rangle = \langle \varphi, \Theta'_\lambda\psi\rangle$.

23.2. The smallest examples

What are these matrices for $\mathrm{U}(1)\times\mathrm{U}(1)$ and $\mathrm{U}(1)\times\mathrm{U}(2)$? By Problem 21.1.4 each class set has one element, and the lift is a single block. Everything is expressed through the Gaussian moments

$$c_a := \sum_{\alpha\in\mathcal{O}} \alpha^a e^{-\pi|\alpha|^2} \qquad (a \ge 0).$$

Let ϖ, ϑ_0 and $F = \sigma^2\wp$ be as in §15.1.

Problem 23.2.1. c_a is real, $c_a = 0$ unless $4 \mid a$, and $c_a = \vartheta_0\,(\varpi/2\pi)^a F^{(a)}(0)$ with $F^{(a)}(0) \in \mathbb{Q}$.

Problem 23.2.2. For $a = 0, 4, 8, 12, 16$ the rational numbers $F^{(a)}(0)$ are 1, 4, 272, -7104, 3203328.

We identify $\mathrm{Sym}^a(\mathbb{C}) = \mathbb{C}e^a$ with $\mathbb{C}$ by $e^a \mapsto 1$, so that the values $\varphi(I_1)$ of forms on $\mathrm{U}(1)$ are numbers.

Problem 23.2.3. For $n = n' = 1$ and $4 \mid a$, $\Theta_{(a)}$ is multiplication by $c_a/4$ on the line $\mathcal{A}_{(a)}(1)$.

For $n = 1$ and $n' = 2$, $\mathrm{Hom}_\mathcal{O}(I_1, I_2) = \mathcal{O}^2$ is the space of columns $x = (x_1, x_2)$.

Problem 23.2.4. For $\varphi \in \mathcal{A}_{(a)}(1)$,

$$(\Theta_{(a)}\varphi)(I_2) = \frac{\varphi(I_1)}{4} \sum_j \binom{a}{j} c_j c_{a-j}\, e_1^j e_2^{a-j}.$$

For $a = 8$ this is $4\varphi(I_1)\,\vartheta_0^2(\varpi/2\pi)^8\big(17e_1^8 + 70e_1^4e_2^4 + 17e_2^8\big)$.

Problem 23.2.5. The lift $\Theta'_{(a)}\colon \mathcal{A}_{(a)}(2) \to \mathcal{A}_{(a)}(1)$ is

$$(\Theta'_{(a)}\psi)(I_1) = \frac{1}{32} \sum_j c_j c_{a-j}\, \psi_j, \qquad \text{where } \psi(I_2) = \sum_j \psi_j\, e_1^j e_2^{a-j}.$$

How large are these entries?

Problem 23.2.6. $\log c_a^2 \le a\log a - a(1 + \log 2\pi) + O(\log a)$, where $\log 0 := -\infty$.

Numerically the bound is attained up to $O(\log a)$: for $a = 40, 80, \dots, 240$ the left side exceeds the right side without the $O(\log a)$ by between 1.7 and 4.8. The scale 2π belongs to the Gaussian; the rational part $F^{(a)}(0)$ of c_a sees instead the period. By Problem 11.3.7, $\varpi^2/2\pi = \Omega/4$ with $\Omega = 2\varpi^2/\pi$, and by Theorem 11.3.1, $\log\Omega = \gamma + \log\pi - L'(1,\chi)/L(1,\chi)$. Combining these with Problems 23.2.1 and 23.2.6 gives

$$\log F^{(a)}(0)^2 \le a\log a - a\Big(1 + \gamma + \log\frac{\pi}{4} - \frac{L'(1,\chi)}{L(1,\chi)}\Big) + O(\log a). \tag{2}$$

This is how Euler's constant and $L'(1,\chi)/L(1,\chi)$ enter the linear term of a size bound.

23.3. The Gaussian norm and the norm of the lift

The size of an entry is one thing; the size of the lift is measured by its operator norm. What is $\|\Theta_{(a)}\|$ in the smallest case?

Let $\mathcal{F}_m$ be the space of entire functions on $\mathbb{C}^m$ with

$$\|f\|^2 := \pi^{-m}\int_{\mathbb{C}^m} |f(z)|^2 e^{-|z|^2}\, dA(z) < \infty.$$

For $m = 1$ this is the norm of Chapter 5, in which the monomials are orthogonal and $\|z^a\|^2 = a!$ (Problem 5.4.4); since the measure and the inner product of two monomials factor over the variables, the monomials z^ν are orthogonal in $\mathcal{F}_m$ with $\|z^\nu\|^2 = \nu!$. By Problems 21.3.2 and 5.4.4, sending e^ν to z^ν identifies $\mathrm{Sym}^a(\mathbb{C}^m)$ with the homogeneous polynomials of degree a in $\mathcal{F}_m$, with $\|e^\nu\|^2 = \|z^\nu\|^2/a!$. In this normalization the size of a derivative is controlled as follows.

Problem 23.3.1. If $\Theta(Z) = \sum_\nu b_\nu Z^\nu$ and $P = \sum_\nu P_\nu x^\nu$, then

$$|P(\partial)\Theta(0)|^2 \le \|P\|_{\mathcal{F}}^2 \sum_{\nu\in\operatorname{supp} P} \nu!\,|b_\nu|^2.$$

What is the squared norm of the lift from U(1) to U(2)?

Problem 23.3.2. For $\Theta_{(a)}\colon \mathcal{A}_{(a)}(1) \to \mathcal{A}_{(a)}(2)$ and $\varphi \neq 0$,

$$\frac{\|\Theta_{(a)}\varphi\|^2}{\|\varphi\|^2} = \frac{1}{128}\sum_{x,y\in\mathcal{O}^2} h(x,y)^a e^{-\pi(|x|^2+|y|^2)} = \frac{1}{128}\sum_j \binom{a}{j} c_j^2 c_{a-j}^2.$$

For $a = 4$ and $a = 8$ it equals $\frac14\vartheta_0^4(\varpi/2\pi)^8$ and $1296\,\vartheta_0^4(\varpi/2\pi)^{16}$.

The middle expression is a theta sum over $\mathrm{Hom}_{\mathcal{O}}(I_1 \oplus I_1, I_2)$, the lattice of pairs (x,y). This is the structure of the Rallis inner product formula [Ra]. More generally, the diagonal kernel of $\Theta'_\lambda\Theta_\lambda$ is a stabilizer-weighted sum over $L' \in \mathrm{Cl}_{n'}$ of theta series on the doubled lattice $\mathrm{Hom}_{\mathcal{O}}(L \oplus L, L')$. In the convergent range the Siegel–Weil formula identifies the corresponding adelic theta integral with an Eisenstein series [We65], and the doubled theta sum is the construction behind the inner product formula of [Ra, §1]. For $4 \mid a$ the source $\mathcal{A}_{(a)}(1)$ is a line, and Problem 23.3.2 computes the scalar directly.

The two smallest lifts satisfy the following norm estimate.

Theorem 23.3.1. *For the lifts $\Theta_{(a)}\colon \mathcal{A}_{(a)}(1) \to \mathcal{A}_{(a)}(n')$ with $n' = 1, 2$, one has $\|\Theta_{(a)}\|^2 = c_a^2/16$ and $\|\Theta_{(a)}\|^2 = \frac{1}{128}\sum_j \binom{a}{j} c_j^2 c_{a-j}^2$ respectively, and in both cases*

$$\log\|\Theta_{(a)}\|^2 \le a\log a - a(1 + \log 2\pi) + O(\log a).$$

So the squared norm of the lift is at most a power of a times one factorial, $\|z^a\|^2 = a!$, times $(2\pi)^{-a}$. Numerically, for $n' = 2$ and $a = 40, 80, \dots, 240$, the left side exceeds the right side without the $O(\log a)$ by an amount growing from 1.5 to 5.1, roughly like $2\log a - 5.8$.

What kind of number is $c_a^2/16$? For $4 \mid n$ the function $\upsilon_n((\alpha)) := \alpha^n$ is well defined on the nonzero ideals of $\mathcal{O}$, since the units of $\mathcal{O}$ are fourth roots of unity; it is a Hecke character. For $\Re s > n/2 + 1$ let

$$L(\bar\upsilon_n, s) := \sum_{\mathfrak{a}\neq 0} \overline{\upsilon_n(\mathfrak{a})}\, \mathrm{N}\mathfrak{a}^{-s} = \frac{1}{4} \sum_{\alpha\in\mathcal{O}}{}' \bar\alpha^n |\alpha|^{-2s}.$$

For $n \geq 4$ the series converges at $s = n$, where it equals $\frac{1}{4}\sum'_\alpha \alpha^{-n} = (2\varpi)^n H_n/(4\cdot n!)$ by Problem 11.3.1.

Problem 23.3.3. For $4 \mid n$ and $n \geq 4$, $\left|\sum'_\alpha \alpha^{-n} - 4\right| < 3$. Hence $L(\bar\upsilon_n, n) > 0$ and $H_n > 0$.

Problem 23.3.4. For $4 \mid a$ and $a \geq 4$ there is $\kappa_a \in \mathbb{Q}$ (a local notation of this chapter) with

$$\frac{c_a^2}{16} = \kappa_a\, \vartheta_0^2\, (2\pi)^{-2a}\, L(\bar\upsilon_{2a}, 2a),$$

and $\kappa_a \neq 0$ if and only if $F^{(a)}(0) \neq 0$. Moreover $\kappa_4 = 2100$ and $\kappa_8 = 22992840000$.

So for $4 \mid a$, $a \geq 4$, the squared norm of the smallest lift is a value of a Hecke L-function times the factor $\vartheta_0^2(2\pi)^{-2a}$, up to a rational factor that the problem does not explain.

23.4. The seesaw

A seesaw computes one period in two ways: lift to a large group and restrict to a subgroup, or restrict first and lift between smaller groups [Ku]. In the definite lattice model, take an orthogonal decomposition $L' = L_1 \perp L_2$.

Problem 23.4.1. $\Theta_{L,L_1\perp L_2}(Z_1, Z_2) = \Theta_{L,L_1}(Z_1)\,\Theta_{L,L_2}(Z_2)$.

Here is the smallest case. Write $I_2 = Y \perp \mathcal{O}z$ with $Y = \mathcal{O}e_1$ and $z = e_2$, and let $0 \leq j \leq a$ be integers with $4 \mid j$ and $4 \mid a - j$. For $\psi \in \mathcal{A}_{(a)}(2)$ and $\xi \in \mathcal{A}_{(j)}(1)$, viewed as a form on the class set of Y, the *period* of ψ along Y is

$$P(\psi, \xi) := \frac{1}{|\Gamma_Y|} \big\langle \psi(Y \perp \mathcal{O}z),\ \xi(Y)\, e_2^{a-j} \big\rangle.$$

On the other side let $\theta_z := \vartheta_{(a-j)}(\mathcal{O}z, I_1)\, e_2^{a-j} \in \mathrm{Sym}^{a-j}(\mathbb{C})$, the vector obtained from $\mathcal{O}z$ alone. Since $4 \mid a$, we identify each vector of $\mathrm{Sym}^a((I_1)_\mathbb{C})$ with the unique form in $\mathcal{A}_{(a)}(1)$ taking that value at I_1.

Problem 23.4.2. For $\varphi \in \mathcal{A}_{(a)}(1)$ and $\xi \in \mathcal{A}_{(j)}(1)$,

$$P(\Theta_{(a)}\varphi, \xi) = \big\langle \varphi,\ (\Theta_{(j)}\xi)(I_1)\, \theta_z \big\rangle,$$

where the pairing on the right is the inner product of $\mathcal{A}_{(a)}(1)$, the product is that of $\mathrm{Sym}^j(\mathbb{C}) \otimes \mathrm{Sym}^{a-j}(\mathbb{C}) \to \mathrm{Sym}^a(\mathbb{C})$, and $e_1 \mapsto 1$ identifies Y with I_1 for the lift of ξ; both factors of the product take values in the symmetric algebra of $(I_1)_\mathbb{C} = \mathbb{C}$. Both sides equal $\varphi(I_1)\overline{\xi(I_1)}\, c_j c_{a-j}/16$.

The left side lifts φ to U(2) and integrates along U(1). The right side lifts ξ to U(1) and pairs with φ there; it depends on ξ only through $\Theta_{(j)}\xi$. Here both sides are computed explicitly.

Problem 23.4.3. For $(a,j) = (8,4)$ both sides of Problem 23.4.2 equal $\varphi(I_1)\overline{\xi(I_1)}\,\vartheta_0^2(\varpi/2\pi)^8$.

In the proof an identity of the same kind holds for chains of m lifts (Theorem 43.4.1): on the ramified source forms, the lower chain applied to the Poincaré image of a form equals one constant $c \neq 0$, independent of the twist, times a composite lift along the upper chain restricted to Y_m, whose data are chosen for each twist.

23.5. Chains

How do kernels and norms behave under composition?

Problem 23.5.1. The composite $\mathcal{A}_\lambda(n) \to \mathcal{A}_\lambda(n') \to \mathcal{A}_\lambda(n'')$ of two lifts is $\varphi \mapsto \big(L'' \mapsto \sum_{L\in \mathrm{Cl}_n} |\Gamma_L|^{-1} K(L,L'')\,\varphi(L)\big)$, with the kernel

$$K(L,L'') = \sum_{L'\in \mathrm{Cl}_{n'}} \frac{1}{|\Gamma_{L'}|}\,\vartheta_\lambda(L',L'')\,\vartheta_\lambda(L,L').$$

Problem 23.5.2. In the bases e^μ of $\mathrm{Sym}^a(\mathbb{C}^n)$ and $\mathrm{Sym}^a(\mathbb{C}^{n'})$, every entry of $\vartheta_{(a)}(I_n, I_{n'})$ lies in $\vartheta_0^{nn'}(\varpi/2\pi)^a\,\mathbb{Q}$.

So for fixed n, n' and a, the entries of the standard kernel have the common factor $\vartheta_0^{nn'}(\varpi/2\pi)^a$. For a term of the composite kernel passing through standard lattices of ranks $n_0,\dots,n_m$, the product of its kernel entries lies in $\vartheta_0^{\sum_{j=1}^m n_{j-1}n_j}(\varpi/2\pi)^{ma}\,\mathbb{Q}$; the class sums add the weights $1/|\Gamma_L|$.

Problem 23.5.3. Let $S \subset R$ be commutative rings and $D_1,\dots,D_m \in R$. Suppose that in a composite of m maps $(T_j f)(i') = \sum_i h_j(i)\,K_j(i,i')\,f(i)$, with finite sums and $h_j(i) \in R$, all $K_j(i,i')$ and all starting values lie in S, and that $D_j h_j(i) \in S$ for all i. Then $D_1\cdots D_m$ times every final value lies in S.

How large is a chain? It is measured by the volume $\det(u^*u)$ of the map u it defines.

Problem 23.5.4. Let $u_j\colon H_{j-1} \to H_j$, $1 \le j \le m$, be linear maps between complex inner product spaces of the same finite dimension, and $u := u_m \circ \cdots \circ u_1$. Then $\det(u^*u) = \prod_j \det(u_j^* u_j)$. For $4 \mid a$ the space $\mathcal{A}_{(a)}(1)$ is a line, and for the chain of m lifts $\Theta_{(a)}\colon \mathcal{A}_{(a)}(1) \to \mathcal{A}_{(a)}(1)$ this is $(c_a^2/16)^m$.

Problem 23.5.5. Let $u(\omega)\colon H \to H'$ be linear maps of finite-dimensional complex inner product spaces, depending on a parameter ω, with $\langle u(\omega)x, u(\omega)y\rangle = a(\omega)\langle S(\omega)x, y\rangle$ for all $x, y \in H$, where $a(\omega) \in \mathbb{C}$ and $S(\omega) \in \mathrm{End}(H)$. Then $\det(u(\omega)^*u(\omega)) = a(\omega)^{\dim H}\det S(\omega)$. Hence, if $\det(u(1)^*u(1)) \neq 0$ and $a(\omega) = c\,b(\omega)$ with c independent of ω and $b(1) \neq 0$, the ratio $\det(u(\omega)^*u(\omega))/\det(u(1)^*u(1))$ equals $(b(\omega)/b(1))^{\dim H}\det S(\omega)/\det S(1)$ and does not involve c.

For the chains of the proof see Chapter 34. Theorem 55.4.1 compares the determinant of the chain from Y_0 to Y_m with the determinants of the single lifts from W_j

to V_j. Problem 23.5.4 gives a model for composition, and Problem 23.5.5 explains how division by the value at the trivial twist cancels the common constant.

Beyond the smallest cases a chain needs more. The class sets grow with the rank (Section 21.2), and in the proof the norm of a lift comes from Rallis's inner product formula (Theorem 56.2.1) in place of Problem 23.3.2. For the moments c_a the rational factors are the derivatives $F^{(a)}(0)$; Problem 15.5.3 gives a one-variable obstruction at inert primes, and Theorem 16.3.1 gives a divisibility bound for the normalized two-variable moments $t(a, w)$ of Section 15.1 when $p \equiv 3 \pmod 8$. Constants need not be exact: one seesaw constant works for all twists (Theorem 43.4.1), and constants independent of the twist cancel in the ratio of Theorem 55.4.1.

In the upper-bound argument of the proof, the seesaw transfers a size bound for the composite lift to the lower chain on Poincaré images (Theorem 43.4.1). In the lower-bound argument, the norms of single lifts enter the factorization of the ratio of chain determinants (Theorem 55.4.1).

In the Lean formalization. The corresponding statement of the Lean proof [GL] for the squared norm of a lift, as in Problem 23.3.4, is `New.LValueData.inner_trivialLift_self_eq_L` in `New/OneLift/HeckeValues/CharacterLiftNorm.lean`; it concerns the lift of a character from U(1) to a definite U($2k$) with $k \geq 2$.

23.6. Questions

For a tending to infinity through multiples of 4, does $\log c_a^2 = a \log a - a(1 + \log 2\pi) + O(\log a)$ hold? Section 23.2 gives numerical examples.

Hints

§23.1.

23.1.1: In bases, $h(x, y) = \sum_{\alpha,\beta} x_{\beta\alpha}\bar{y}_{\beta\alpha}$ when L and L' are standard. In general, if G and G' are the Gram matrices of L and L', so that $h(u, v) = \bar{v}^t G u$, then $y^* = G^{-1}\bar{y}^t G'$; compute the Gram matrix of $\mathrm{Hom}(L, L')$ and its determinant.

23.1.2: Differentiate under the sum; $\partial/\partial Z_{\alpha\beta}$ brings down $x_{\beta\alpha}$.

23.1.3: The entries of $x^{\otimes k}$ are monomials in the entries of x. For the vanishing, replace x by $\mathrm{i}x$.

23.1.4: Substitute $x \mapsto \gamma' x$ and $x \mapsto x\gamma$; both preserve $\mathrm{Hom}_{\mathcal{O}}(L, L')$ and $h(x, x)$.

23.1.5: $\mathbb{S}_\lambda(x)^* = \mathbb{S}_\lambda(x^*)$. What does $x \mapsto x^*$ do to $\mathrm{Hom}(L, L')$ when L and L' are unimodular?

§23.2.

23.2.1: Use the symmetries of the square lattice and Problem 15.1.3. For rationality: the Laurent coefficients of $\wp$ lie in $\mathbb{Q}[g_2, g_3]$, and $(\log\sigma)'' = -\wp$.

23.2.2: Expand σ and $\wp$ with $g_2 = 4$, $g_3 = 0$ further than in Problem 15.1.2.

23.2.3: $\Gamma_{I_1} = \mu_4$, and $\mathrm{Sym}^a(x)$ is multiplication by x^a.

23.2.4: $\mathrm{Sym}^a(x)$ sends $1 \in \mathrm{Sym}^a(\mathbb{C})$ to $(x_1 e_1 + x_2 e_2)^a$. Expand by the binomial theorem; the sum over $\mathcal{O}^2$ factors.

23.2.5: $\mathrm{Hom}(I_2, I_1)$ consists of rows y, and $\mathrm{Sym}^a(y)\, e_1^j e_2^{a-j} = y_1^j y_2^{a-j}$. Alternatively use Problems 23.1.5 and 21.3.2.

23.2.6: $|c_a| \le \sum_\alpha f(\alpha)$ with $f(z) := |z|^a e^{-\pi|z|^2}$. Bound the $O(a)$ terms with $|\alpha|^2 \le a$ by $\max f$, attained at $|z|^2 = a/2\pi$, and the others by a geometric series.

§23.3.

23.3.1: Problem 23.1.2, and the Cauchy–Schwarz inequality.

23.3.2: $\|\Theta\varphi\|^2 = \langle \Theta'\Theta\varphi, \varphi\rangle$; use Problems 23.2.4 and 23.2.5. For the middle expression use $\langle u^a, v^a\rangle = \langle u, v\rangle^a$.

Theorem 23.3.1: Combine Problems 23.2.3, 23.2.6 and 23.3.2 with $\binom{a}{j} j!(a-j)! = a!$ and Stirling's formula.

23.3.3: Separate the units, and compare the other terms with Problem 11.3.4 at $s = 2$.

23.3.4: Problem 23.2.1, the expression of $L(\bar{v}_n, n)$ by H_n, and Problem 23.3.3. For κ_8: $H_{16} = 43659/170$.

§23.4.

23.4.1: How does $\mathrm{Hom}_{\mathcal{O}}(L, L_1 \perp L_2)$ decompose, and what happens to h and $\mathrm{tr}(Zx)$?

23.4.2: Problems 23.2.3, 23.2.4 and 21.3.2.

23.4.3: Problems 23.2.1 and 15.1.2.

§23.5.

23.5.1: Substitute (1) into itself.

23.5.2: Expand a single monomial coefficient of $\mathrm{Sym}^a(x)\, e^\mu$, sum it over x, and compare with Problem 23.2.1.

23.5.3: Induction on m: move D_1 inside the first sum.

23.5.4: In orthonormal bases $\det(u^* u) = |\det u|^2$. For the lifts use Problem 23.2.3.

23.5.5: Express the Gram identity through the adjoint.

References

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[Ku] S. S. Kudla, *Seesaw dual reductive pairs*, in *Automorphic Forms of Several Variables (Katata, 1983)*, Progr. Math. 46, Birkhäuser, 1984, 244–268.

[Ra] S. Rallis, *Injectivity properties of liftings associated to Weil representations*, Compositio Math. **52** (1984), 139–169.

[We65] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.