MASSES AND THE COUNT OF FORMS

Prerequisites. Hermitian forms over the Gaussian integers, at the level of definitions. The classification of positive definite unimodular $\mathbb{Z}$-lattices of rank at most 8 [CoS, Ch. 16]. Schur functors and the Weyl dimension formula for $\mathrm{GL}_n(\mathbb{C})$ [FH, Theorem 6.3 and Eq. (15.17)]. The values of Dirichlet L-functions at negative integers [Wa, Theorem 4.2], and the functional equations of $\zeta(s)$ and $L(s,\chi)$ [Dav, Ch. 8, 9]. Stirling's formula. No earlier chapter is used.

Chapter 8 introduces a chain of linear maps. Chapter 23, *Definite theta lifts and their norms*, studies theta lifts between definite unitary groups; the corresponding construction of the proof is the composite lift of Chapter 33. For the definite spaces we take definite unitary groups over $\mathbb{Z}[\mathrm{i}]$ as a model. The genus of the standard lattice has finitely many classes. At the level given by the stabilizer of that lattice, a form of a fixed weight assigns an invariant vector in the weight representation to each class. How many forms are there? For fixed positive rank, when 4 divides the total degree of the weight and its gaps are large, the answer is asymptotically four times the product of a mass and a Weyl dimension, and both factors carry $\prod_{j<n} j!$, which cancels. Chapter 22 evaluates examples of genus averages and relates them to a rank-one lift norm.

21.1. Hermitian lattices over $\mathbb{Z}[\mathrm{i}]$

Let $\mathcal{O} := \mathbb{Z}[\mathrm{i}]$ and $K := \mathbb{Q}(\mathrm{i})$. A *Hermitian lattice* of rank n is a free $\mathcal{O}$-module L of rank n with a positive definite Hermitian form $h\colon L \times L \to \mathcal{O}$, linear in the first variable. Its dual is $L^\sharp := \{y \in L \otimes_{\mathcal{O}} K : h(x,y) \in \mathcal{O} \text{ for all } x \in L\}$, and L is *unimodular* if $L^\sharp = L$. The basic example is $I_n := \mathcal{O}^n$ with $h(x,y) = \sum_k x_k \bar{y}_k$. Let Γ_L be the group of $\mathcal{O}$-linear automorphisms of L preserving h; it is finite, since it permutes the finitely many vectors of bounded length that span L.

Two Hermitian lattices are in the same *genus* if they become isometric over $\mathcal{O} \otimes \mathbb{Z}_p$ for every prime p. Let Cl_n be the set of isometry classes in the genus of I_n. By Problem 21.1.1 below, a class gives a unimodular $\mathbb{Z}$-lattice of rank $2n$ together with an automorphism of it; there are finitely many such lattices up to isometry [Ca, Ch. 9], each with a finite automorphism group, so Cl_n is finite. We call Cl_n the *class set*. For the unitary group of I_n it is the finite adelic double-coset set at the stabilizer of the standard lattice. Its *mass* is

$$M_n := \sum_{L \in \mathrm{Cl}_n} \frac{1}{|\Gamma_L|}, \qquad M_0 := 1.$$

What do the class sets look like in small rank?

Problem 21.1.1. Let L be a Hermitian lattice and $B(x,y) := \operatorname{Re} h(x,y)$. Then B is a positive definite symmetric $\mathbb{Z}$-bilinear form with $B(x,x) = h(x,x)$, multiplication by i preserves B, and $h(x,y) = B(x,y) + \mathrm{i}B(x,\mathrm{i}y)$. The dual of L for B is $L^\sharp$; in particular L is unimodular exactly when (L,B) is.

Date: November 25, 2026.

Problem 21.1.2. Γ_{I_n} is the group of monomial matrices whose nonzero entries lie in $\mu_4 = \{\pm 1, \pm \mathrm{i}\}$. Its order is $4^n n!$.

Problem 21.1.3. For $n \geq 1$, every lattice in the genus of I_n contains a vector x with $h(x,x)$ odd.

Problem 21.1.4. If $n \leq 4$, then $\mathrm{Cl}_n = \{I_n\}$.

Combining Problems 21.1.2 and 21.1.4 gives $M_n = 1/(4^n n!)$ for $n \leq 4$. Let Λ be the Gaussian Hermitian lattice with underlying real lattice E_8. Its norms are even, and $|\Gamma_\Lambda| = 46080$, the order of the reflection group G_{31} of Shephard and Todd [ShT].

21.2. The mass of a genus

How does M_n grow? Let χ be the non-trivial character modulo 4, and let χ^j be χ for odd j and the trivial character for even j, so that $L(s, \chi^j) = \zeta(s)$ for even j. For the genus of I_n, the mass formula reads

$$M_n = \delta_n \prod_{j=1}^{n} \frac{|L(1-j, \chi^j)|}{2}, \tag{1}$$

where δ_n is a positive rational number, a local factor at the prime 2, which is ramified in K. This is the Hermitian analogue of Siegel's mass formula for quadratic forms [Si], with the local factors computed by Gan and Yu [GY] and, at the ramified prime 2, by Cho [Cho]. We set $\delta_0 := 1$, so that (1) also holds for $n = 0$, where $M_0 = 1$.

Problem 21.2.1. For $j = 1, 2, 3, 4, 5$, the numbers $|L(1-j, \chi^j)|/2$ are 1/4, 1/24, 1/4, 1/240, 5/4.

Problem 21.2.2. Determine the local factors from the class masses, taking the local factors as unknowns. Prove that $\delta_1 = \delta_3 = 1$, $\delta_2 = 3$ and $\delta_4 = 15$.

For all positive ranks, [JZ, Corollary 4.3] gives $\delta_n = 1$ for odd n and $\delta_n = 2^n - 1$ for even n.

In rank 5, with $\delta_5 = 1$, the formula gives $M_5 = 1/73728$. Two unimodular lattices of the same rank over $\mathcal{O}$, each containing a vector x with $h(x,x)$ odd, lie in the same genus [JZ, Corollary 2.2]; thus $\Lambda \perp I_1 \in \mathrm{Cl}_5$.

Problem 21.2.3. The only vectors $x \in \Lambda \perp I_1$ with $h(x,x) = 1$ are the four multiples ue, $u \in \mu_4$, of a generator e of I_1; hence $\Gamma_{\Lambda \perp I_1} = \Gamma_\Lambda \times \mu_4$. Moreover $\mathrm{Cl}_5 = \{I_5, \Lambda \perp I_1\}$.

What is the factorial contribution of the jth mass factor?

Problem 21.2.4. For $j \geq 1$,

$$\frac{|L(1-j, \chi^j)|}{2} = (j-1)!\, \pi^{-j} \beta_j, \qquad \beta_j := \begin{cases} 2^{j-1} L(j, \chi) & j \text{ odd}, \\ 2^{-j} \zeta(j) & j \text{ even}. \end{cases}$$

For $r \geq 1$ define $\kappa_r > 0$ by

$$\frac{M_r}{M_{r-1}} = (r-1)!\, \pi^{-r} \kappa_r. \tag{2}$$

By (1) and Problem 21.2.4, $\kappa_r = (\delta_r/\delta_{r-1})\beta_r$. Multiplying (2) over $r \le n$,

$$M_n = \pi^{-n(n+1)/2} \prod_{j=0}^{n-1} j! \prod_{j=1}^{n} \kappa_j. \tag{3}$$

After removing $(r-1)!$ from the mass ratio, the logarithm of the remaining factor is $-r\log\pi + \log\kappa_r$.

Problem 21.2.5. For $r \le 4$, $\kappa_r = \pi^r/(4\cdot r!)$; thus $\kappa_1, \dots, \kappa_4$ are $\pi/4$, $\pi^2/8$, $\pi^3/24$, $\pi^4/96$. With $\delta_5 = 1$, $\kappa_5 = \pi^5/288$.

Problem 21.2.6. For every $r \ge 1$, $3/4 \le \kappa_r \le 2$; in particular $\log\kappa_r = O(1)$.

By (3), $\log M_n$ grows like $\frac{1}{2}n^2\log n$; for $n \ge 1$ every Γ_L contains μ_4, so the class set has at least $4M_n$ elements, more than $8\cdot 10^5$ for $n = 15$.

21.3. Forms and the Weyl dimension formula

On a definite group a form is a function on the class set, with values in a representation. How many are there?

For $L \in \mathrm{Cl}_n$ let $L_{\mathbb{C}} := L \otimes_{\mathcal{O}} \mathbb{C}$, using the inclusion $\mathcal{O} \subset \mathbb{C}$. The form h extends to a Hermitian inner product on $L_{\mathbb{C}}$, and Γ_L is a finite subgroup of its unitary group. A *weight* is a sequence of integers $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$; write $|\lambda| := \sum_k \lambda_k$. For a finite-dimensional Hermitian space A, let $\mathbb{S}_\lambda(A) \subset A^{\otimes|\lambda|}$ be the image of the Young symmetrizer [FH, §6.1], with the inner product induced from $A^{\otimes|\lambda|}$. A linear map $x\colon A \to B$ induces $\mathbb{S}_\lambda(x)\colon \mathbb{S}_\lambda(A) \to \mathbb{S}_\lambda(B)$, the restriction of $x^{\otimes|\lambda|}$. Then $\mathbb{S}_\lambda(xy) = \mathbb{S}_\lambda(x)\mathbb{S}_\lambda(y)$ and $\mathbb{S}_\lambda(x)^* = \mathbb{S}_\lambda(x^*)$, the latter because $(x^*)^{\otimes|\lambda|}$ intertwines the permutation actions on the two tensor powers. If λ has at most $\dim A$ nonzero entries, $\mathbb{S}_\lambda(A)$ is the irreducible representation of $\mathrm{GL}(A)$ of highest weight λ; otherwise it is zero [FH, Theorem 6.3]. For $\lambda = (a)$ it is $\mathrm{Sym}^a(A)$. There, for $u \in A$ we write $u^a := u \otimes \cdots \otimes u$; for a basis $e_1, \dots, e_n$ of A and $|\nu| = a$, the vectors e^ν are defined by $(\sum_k t_k e_k)^a = \sum_\nu (a!/\nu!)\, t^\nu e^\nu$, and $e^\nu e^\mu := e^{\nu+\mu}$ is the product of the symmetric algebra.

Let $V_\lambda(L) := \mathbb{S}_\lambda(L_{\mathbb{C}})$. Weyl's dimension formula reads, with $z_k := \lambda_k + n - k$,

$$\dim V_\lambda(L) = D_n(z) := \frac{V_n(z)}{\prod_{j=0}^{n-1} j!}, \qquad V_n(z) := \prod_{1\le k<l\le n} (z_k - z_l) \tag{4}$$

[FH, Eq. (15.17)]. Fix one lattice in each class of Cl_n. A *form of weight* λ assigns to each $L \in \mathrm{Cl}_n$ a vector $\varphi(L) \in V_\lambda(L)$ fixed by Γ_L. The forms make up a space $\mathcal{A}_\lambda(n)$, with the squared norm

$$\|\varphi\|^2 := \sum_{L\in\mathrm{Cl}_n} \frac{1}{|\Gamma_L|}\, \|\varphi(L)\|^2.$$

These are the algebraic modular forms of Gross [Gr] on the unitary group of I_n, of level one.

Problem 21.3.1. For $n \ge 1$ and $a \ge 0$, $\dim \mathrm{Sym}^a(\mathbb{C}^n) = \binom{a+n-1}{n-1}$, and $\dim \mathbb{S}_{(3,2,0)}(\mathbb{C}^3) = 15$. The denominator in (4) is $\prod_{j=0}^{n-1} j! = V_n(n-1, \dots, 1, 0)$.

Problem 21.3.2. In $\mathrm{Sym}^a(A)$, $\langle u^a, v^a\rangle = \langle u, v\rangle^a$. If $e_1, \dots, e_n$ is orthonormal, the e^ν are orthogonal and $\|e^\nu\|^2 = \nu!/a!$.

Problem 21.3.3. $\mathcal{A}_\lambda(n) = 0$ unless $4 \mid |\lambda|$. The space $\mathcal{A}_{(a)}(1)$ is one-dimensional when $4 \mid a$.

Problem 21.3.4. $\dim \mathcal{A}_\lambda(n) = \sum_L |\Gamma_L|^{-1} \sum_{\gamma \in \Gamma_L} \operatorname{tr}(\gamma \mid V_\lambda(L))$. For $n \geq 1$ and $4 \mid |\lambda|$, the scalars $\gamma \in \mu_4$ contribute $4M_n \dim V_\lambda$ to this sum.

Problem 21.3.5. The Γ_{I_2}-invariants in the symmetric algebra of $\mathbb{C}^2$ form the polynomial algebra on $e_1^4 + e_2^4$ and $e_1^4 e_2^4$. Hence, for $4 \mid a$, $\dim \mathcal{A}_{(a)}(2) = \lfloor a/8 \rfloor + 1$, which is $(1 + o(1)) \cdot 4M_2 \dim V_{(a)}$ as $a \to \infty$.

Does $|\lambda| \to \infty$ suffice? Compare two weights of the same size.

Problem 21.3.6. Let $j \geq 1$, $\lambda := (8j, 0)$ and $\lambda' := (4j, 4j)$. Then $\dim \mathcal{A}_\lambda(2) = j+1$ and $\dim \mathcal{A}_{\lambda'}(2) = 1$, while the scalar contributions $4M_2 \dim V$ of Problem 21.3.4 are $j + \frac{1}{8}$ and $\frac{1}{8}$. In both cases the non-scalar elements of Γ_{I_2} contribute exactly $\frac{7}{8}$.

These weights have the same total degree and different gaps.

Fix a positive rank n. When $4 \mid |\lambda|$ and all the gaps $\lambda_k - \lambda_{k+1}$ tend to infinity, the Weyl character formula shows that each non-scalar γ contributes $o(\dim V_\lambda)$, so $\dim \mathcal{A}_\lambda(n) \sim 4M_n \dim V_\lambda$. The count of forms is thus asymptotically four times the product of the mass and the Weyl dimension. By (3) and (4) each factor contains $\prod_{j<n} j!$, once in the numerator and once in the denominator; §21.4 returns to this.

21.4. The count of forms

When $4 \mid |\lambda|$ and the gaps of λ are large, the space of forms of weight λ has dimension about $4M_n \dim V_\lambda$ (§21.3). The mass contains the product $\prod_{j<n} j!$, whose logarithm is about $\frac{1}{2}n^2 \log n$, and the Weyl dimension contains its inverse. How does $M_r D_r$ change when the rank increases?

Problem 21.4.1. $\sum_{j=0}^{n-1} \log j! = \frac{1}{2}n^2 \log n - \frac{3}{4}n^2 + O(n \log n)$.

Problem 21.4.2. Let $r_0 < r_1$, and let $u \in \mathbb{Z}^{r_0}$ and $w \in \mathbb{Z}^{r_1}$ be strictly decreasing. Then

$$\frac{M_{r_1} D_{r_1}(w)}{M_{r_0} D_{r_0}(u)} = \frac{V_{r_1}(w)}{V_{r_0}(u)} \, \pi^{-\sum_{r=r_0+1}^{r_1} r} \prod_{r=r_0+1}^{r_1} \kappa_r.$$

With $r_0 = 0$, where $M_0 = D_0 = V_0 = 1$ (empty products), Problem 21.4.2 reads as follows.

Theorem 21.4.1 (The count identity)**.** *For every $r \geq 1$ and every strictly decreasing $z \in \mathbb{Z}^r$,*

$$M_r D_r(z) = V_r(z) \, \pi^{-r(r+1)/2} \prod_{j=1}^{r} \kappa_j.$$

No factorial survives: what remains is a product of differences of weights, a power of π and the factors κ_j.

The simplest instance adds one rank, with one large new weight.

Problem 21.4.3. Fix $r_0 \geq 1$ and a strictly decreasing $u \in \mathbb{Z}^{r_0}$. For an integer $w_1 > u_1$, let $w = (w_1, u_1, \ldots, u_{r_0})$. Then

$$\log \frac{M_{r_0+1} D_{r_0+1}(w)}{M_{r_0} D_{r_0}(u)} = \sum_{i=1}^{r_0} \bigl(\log(w_1 - u_i) - \log \pi\bigr) - \log \pi + \log \kappa_{r_0+1}.$$

For fixed r_0 and u, the sum in this formula grows like $r_0 \log w_1$ as $w_1 \to \infty$, and the remaining term $\log \kappa_{r_0+1} - \log \pi$ is constant.

In the Lean formalization. This chapter serves the bound for the dimension d_T of the target forms T of Chapter 33, which the Lean proof [GL] combines with a second-moment estimate in `targetDim_mul_sum_hilbertSchmidtNorm_sq_le`. For parameters and data as in Chapter 33, with $\bar{G}$ and the definite space W_m as there, `finrank_principalForms_spaceW` states that the complex dimension of the principal forms $H(W_m)$ equals $\#\bar{G}\cdot$`mass D.E D.r`$\cdot\prod_v$ `(toHCParam 1 (D.paramW v)).formalDegree`, the product over the infinite places v; and `targetDim_le_card_mul_besselDegree_mul_mass` states $d_T \le \#\bar{G} \cdot$ `D.filling.besselDegree` $\cdot$ `mass D.E D.r`. Both are in `New/Spanning/SecondMoment/SpanningTheorem.lean` of the readable edition.

Hints

§21.1.

21.1.1: Compute $B(\mathrm{i}x, \mathrm{i}y)$ and $B(x, \mathrm{i}y)$. For the dual, replace x by $\mathrm{i}x$ in the condition $B(x,y) \in \mathbb{Z}$.

21.1.2: An automorphism permutes the vectors with $h(x,x) = 1$. Which vectors of I_n are these?

21.1.3: Over $\mathbb{Z}_2$ the lattice is isometric to $I_n \otimes \mathbb{Z}_2$. Approximate a vector of odd length 2-adically by a vector of L.

21.1.4: Use Problems 21.1.1 and 21.1.3 and the classification of odd unimodular lattices of rank at most 8; then look at the vectors of length one.

§21.2.

21.2.1: For odd j use $B_{j,\chi} = 4^{j-1}\sum_{a=1}^{4}\chi(a)B_j(a/4)$. For even j use $\zeta(1-j) = -B_j/j$.

21.2.2: Compare (1) with $M_n = 1/(4^n n!)$.

21.2.4: For ζ use $\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s)$. For the odd character χ of conductor 4, whose Gauss sum is 2i, complete $L(s,\chi)$ by $(4/\pi)^{(s+1)/2}\Gamma(\frac{s+1}{2})$, and use the reflection and duplication formulas for Γ. Check the case $j = 1$: $L(1,\chi) = \pi/4$.

21.2.3: Λ is even. For the second part, compare with the mass M_5.

21.2.5: Use $M_r/M_{r-1} = 1/(4r)$ for $r \le 4$, and $M_5 = 1/73728$.

21.2.6: Use $\kappa_r = (\delta_r/\delta_{r-1})\beta_r$ and the values of δ_n, treating odd and even r separately; $1 - 3^{-r} \le L(r,\chi) \le 1$. For $r = 1$ use Problem 21.2.5.

§21.3.

21.3.1: For $\lambda = (a)$, $z = (a+n-1, n-2, \dots, 0)$.

21.3.2: Expand $\langle(\sum t_k e_k)^a, (\sum s_k e_k)^a\rangle$ in two ways.

21.3.3: The scalar $\mathrm{i} \in \Gamma_L$ acts on $V_\lambda(L)$ by $\mathrm{i}^{|\lambda|}$.

21.3.4: The dimension of the invariants of a finite group is the average of the trace.

21.3.5: An invariant must be invariant under $e_k \mapsto \mathrm{i}e_k$ and under the swap. Count the pairs (i,j) with $4i + 8j = a$.

21.3.6: For λ use Problem 21.3.5. For λ', $\mathbb{S}_{(k,k)}(\mathbb{C}^2)$ is the k-th power of the determinant, and every element of Γ_{I_2} has determinant in μ_4. Use $M_2 = 1/32$; the non-scalar part is the dimension less the scalar contribution.

§21.4.

21.4.1: For $j \ge 1$, $\log j! = j\log j - j + O(\log(j+1))$; compare the sum with an integral. The $j = 0$ term vanishes.

21.4.2: Write the left side as a telescoping product, and use (2) and (4).
21.4.3: $V_{r_0+1}(w) = V_{r_0}(u) \prod_i (w_1 - u_i)$.

References

[Ca] J. W. S. Cassels, *Rational Quadratic Forms*, Academic Press, 1978.

[Cho] S. Cho, *Group schemes and local densities of ramified hermitian lattices in residue characteristic 2*, arXiv:1210.7894v1, 2012.

[CoS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Grundlehren Math. Wiss. 290, Springer, 1999.

[Dav] H. Davenport, *Multiplicative Number Theory*, 3rd ed., Grad. Texts in Math. 74, Springer, 2000.

[FH] W. Fulton and J. Harris, *Representation Theory: A First Course*, Grad. Texts in Math. 129, Springer, 1991.

[GY] W. T. Gan and J.-K. Yu, *Group schemes and local densities*, Duke Math. J. **105** (2000), 497–524.

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[Gr] B. H. Gross, *Algebraic modular forms*, Israel J. Math. **113** (1999), 61–93.

[ShT] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.

[JZ] M. Jürgens and M. C. Zimmermann, *Steinitz classes and partial genera of unimodular lattices over imaginary-quadratic fields*, arXiv:1512.03297, 2015.

[Si] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. **36** (1935), 527–606.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Grad. Texts in Math. 83, Springer, 1997.