INERT PRIMES AND THE CHOICE OF d

Prerequisites. Chapter 1, *From zeros to a one-sided bound*, Sections 1.1, 1.2 and 1.4: the functions G and G_χ of its equations (1) and (3), the transform F of (2), Landau's theorem and the argument of Problem 1.2.4. Quadratic reciprocity with its supplements [SeA, Ch. I, §3], the Jacobi symbol and its reciprocity law [IR, Ch. 5], and the splitting of primes in an imaginary quadratic field [Ne, Ch. I, §8]. From Davenport [Dav]: Dirichlet's theorem in the form that $\sum \log p/p$ over the primes of an invertible class diverges (Ch. 4), primitive characters (Ch. 5), the theta functions of a primitive character and their transformation formula (Ch. 9), and the Pólya–Vinogradov inequality (Ch. 23). The three-lines theorem [Ru, Ch. 12].

The criterion of Theorem 1.4.1 assumes a one-sided bound for a remainder of a sum with coefficients $\Lambda(n)(1-\chi_d(n))$ over the $n \equiv 1 \pmod{q_0}$. We first study the unrestricted sum with these coefficients, which are supported on odd powers of primes inert in an imaginary quadratic field and on powers of the ramified prime. How does its centered value compare with G, and what information does it give about the zeros of ζ?

2.1. Inert primes and a quadratic character

Let $d \equiv 3 \pmod 4$ be a prime, let $K := \mathbb{Q}(\sqrt{-d})$, and let $\chi_d(n) := \left(\frac{n}{d}\right)$ be the Legendre symbol.

Problem 2.1.1. (i) For every prime p, $\chi_d(p)$ is 1, -1 or 0 according as p splits, is inert, or ramifies in K. (ii) For a prime power $r = p^h$, $1-\chi_d(r)$ is 2 if p is inert and h is odd, 1 if $p = d$, and 0 otherwise.

The ramp sum over odd powers of inert primes, with weight 2, and over powers of d, with weight 1, is therefore

$$S_d(x) := \sum_r \Lambda(r)\big(1-\chi_d(r)\big)\Big(\frac{x}{r}-1\Big)_+.$$

For a primitive nonprincipal character χ let $\Psi_\chi(t) := \sum_{n\le t}\Lambda(n)\chi(n)$, and let c_χ and G_χ be as in (3).

Problem 2.1.2. $G_\chi(x) = -\sum_n \Lambda(n)\chi(n)\min(1, x/n) = -x\int_x^\infty \Psi_\chi(t)\,t^{-2}\,dt$.

Problem 2.1.3. $S_d(x) - x\log x + (1+\gamma-c_{\chi_d})x = G(x) - G_{\chi_d}(x)$.

An upper bound for S_d minus its main term is thus an upper bound for $G - G_{\chi_d}$, not for G. How much does such a bound say about the zeros of ζ?

Problem 2.1.4. Let $0 < \theta < 1$. Suppose that $L(s,\chi_d)$ has no real zero in $(\theta, 1)$, and that $G(x) - G_{\chi_d}(x) \le x^\theta$ for all large x. Then every zero of $\zeta(s)$ or of $L(s,\chi_d)$ in $\operatorname{Re} s > \theta$ is a zero of both, with the same multiplicity.

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Problem 2.1.4 leaves the possibility of a shared zero of equal multiplicity. We next consider a region free of zeros of $L(s,\chi_d)$. For $\beta, t_0 \in \mathbb{R}$ let

$$\mathcal{B}(\beta,t_0) := \{s \in \mathbb{C} : \operatorname{Re} s \geq \beta,\ |\operatorname{Im} s| \leq t_0\},$$

which is empty for $t_0 < 0$ and unbounded to the right otherwise.

Problem 2.1.5. Let $0 < \theta < \beta$ and $t_0 \geq 0$. Suppose that $L(s,\chi_d) \neq 0$ on $\mathcal{B}(\beta,t_0)$, and that $G(x) - G_{\chi_d}(x) \leq x^\theta$ for all large x. Then ζ has no zero in $\mathcal{B}(\beta,t_0)$.

Fix $\theta > 0$. If the bound $G - G_{\chi_d} \leq x^\theta$ holds for every prime d of some class, and every box $\mathcal{B}(\beta,t_0)$ with $\beta > \theta$ is free of zeros of $L(s,\chi_d)$ for at least one d of the class, then ζ has no zero in $\operatorname{Re} s > \theta$. The choice of d depends on the box. Section 2.2 shows that such a d exists in the class $d \equiv 7 \pmod 8$ when $\theta \geq \frac{1}{2}$. This class is used for the following reason.

Problem 2.1.6. Let $d \equiv 7 \pmod 8$. Then $\chi_d(2) = 1$ and $\chi_d(-1) = -1$. Hence 2 splits in K, $1 - \chi_d(r)$ vanishes at every power of 2, and χ_d is an odd character.

In the proof the weight is restricted further, to prime powers $n \equiv 1 \pmod{q_0}$, and the characters that occur are all ψ modulo q_0 and their twists $\psi\chi_d$. Its criterion, Theorem 1.4.1, takes q_0 a positive multiple of 8, a nonnegative integer $d_0 \equiv 7 \pmod 8$ prime to q_0, and $\beta > \frac{1}{2}$, and assumes, for every prime $d \equiv d_0 \pmod{q_0}$, that the remainder $R_{d,q_0}(x)$ of (4) is at most $x^{\beta-1}$ for all sufficiently large x. Given $\rho \neq 1$ with $\operatorname{Re}\rho > \beta$, its proof uses one such d, given by Theorem 2.2.2 below at the same β, with $N = q_0$, $a = d_0$ and $t_0 = |\operatorname{Im}\rho|$.

2.2. The choice of d

Is there a prime $d \equiv 7 \pmod 8$ for which $L(s,\chi_d)$ has no zero in a given box $\mathcal{B}(\beta,t_0)$ with $\beta > \frac{1}{2}$? Problem 2.2.13 bounds the number of exceptional primes $d \equiv 7 \pmod 8$ up to X. The quantity that detects a zero is built from theta series.

For $j \in \{0,1\}$ let $w_j(z) := z^j e^{-\pi z^2}$ for $z \in \mathbb{R}$, and let

$$\Gamma_j(s) := \int_0^\infty w_j(z)\, z^s\, \frac{dz}{z} \qquad (\operatorname{Re} s > 0).$$

For a sequence $c\colon \mathbb{Z}_{\geq 1} \to \mathbb{C}$ with $|c(n)| \leq 1$ let

$$\Theta_c(u) := u^{1/2} \sum_{n\geq 1} c(n)\, w_j(nu) \qquad (u > 0),$$

and write $c\mu$ for the sequence $n \mapsto c(n)\mu(n)$, with μ the Möbius function. The index j is fixed in each argument and is dropped from Θ_c. A Dirichlet character χ modulo q is a sequence of this kind, and has *parity* j if $\chi(-1) = (-1)^j$.

Problem 2.2.1. $\Gamma_j(s) = \frac{1}{2}\pi^{-(s+j)/2}\,\Gamma\big(\frac{s+j}{2}\big)$ for $\operatorname{Re} s > 0$. In particular $\Gamma_1(1) = 1/(2\pi)$.

Problem 2.2.2. (i) Let χ be a nonprincipal character modulo q of parity j. For $\operatorname{Re} s > 0$ the integral $\int_0^\infty \Theta_\chi(u)\, u^{s-1/2}\, du/u$ converges absolutely and equals $\Gamma_j(s)L(s,\chi)$. For $\chi = \chi_7$, $j = 1$ and $s = 1$ its value is $1/(2\sqrt{7}) = 0.18898\ldots$ (ii) For every character χ and $\operatorname{Re} s > 1$, $\int_0^\infty \Theta_{\chi\mu}(u)\, u^{s-1/2}\, du/u = \Gamma_j(s)/L(s,\chi)$.

Problem 2.2.3. Let χ be a primitive nonprincipal character modulo q of parity j. Then $|\Theta_\chi(1/(qu))| = |\Theta_\chi(u)|$ for $u > 0$. Hence the integrals of $|\Theta_\chi(u)|\, du/u$ over $u < q^{-1/2}$ and over $u > q^{-1/2}$ are equal.

For $T > 0$ define the *tail integral* of a sequence c and the *mollifier* of a character χ:

$$J(c,T) := \int_{1/T}^{\infty} \big(|\Theta_c(u)| + |\Theta_{c\mu}(u)|\big)\,\frac{du}{u}, \qquad \mathcal{M}_T(s) := \int_{1/T}^{\infty} \Theta_{\chi\mu}(u)\, u^{s-1/2}\,\frac{du}{u}.$$

By Problem 2.2.2(ii), for $\operatorname{Re} s > 1$, $\mathcal{M}_T(s)$ is the integral for $\Gamma_j(s)/L(s,\chi)$ cut at $u = 1/T$. For $\operatorname{Re} z > 0$ where $L(z,\chi)$ is finite, let $\Phi(z) := \Gamma_j(z)L(z,\chi)\mathcal{M}_T(z) - \Gamma_j(z)^2$.

Problem 2.2.4. (i) $\sum_{n\ge1}|w_j(nu)| \le 3e^{-u}/u$ for $u > 0$. (ii) For every character χ, every $T > 0$ and $\operatorname{Re} z = 2$: $|\Gamma_j(z)/L(z,\chi) - \mathcal{M}_T(z)| \le 3/T$ and $|\Gamma_j(z)L(z,\chi)| \le 3$, so $|\Phi(z)| \le 9/T$.

Problem 2.2.5. Let χ be a primitive nonprincipal character modulo q of parity j, and $T > 0$ with $q \le T^2$. On $\operatorname{Re} z = \frac12$: $|\Gamma_j(z)L(z,\chi)| \le 2J(\chi,T)$ and $|\mathcal{M}_T(z)| \le J(\chi,T)$.

Problem 2.2.6. There is a constant C_j with the following property. Let $\delta > 0$ and $T > 0$, and let χ be a primitive nonprincipal character modulo q of parity j, with $q \le T^2$ and $J(\chi,T) \le T^{2\delta}$. If $L(s,\chi) = 0$ and $\frac12 + 9\delta \le \operatorname{Re} s \le 1$, then $|\Gamma_j(s)|^2 \le C_j T^{-2\delta}$.

Problem 2.2.7. Let $\delta > 0$ and $t_0 \in \mathbb{R}$. For all T large in terms of δ and t_0 the following holds for every q and every primitive nonprincipal character χ modulo q of parity j with $q \le T^2$: if $L(s,\chi)$ vanishes somewhere on $\mathcal{B}(\frac12 + 9\delta, t_0)$, then $J(\chi,T) > T^{2\delta}$.

A zero in the box therefore forces a large tail integral. It remains to show that the tail integrals of the characters χ_d are small on average over d. Write $n^\circ := n/2^{v_2(n)}$ for the odd part of $n \ge 1$, and $\left(\frac{b}{m}\right)$ for the Jacobi symbol, $m \ge 1$ odd.

Problem 2.2.8. Let $d \equiv 7 \pmod 8$ be prime. For every $n \ge 1$, $\left(\frac{n}{d}\right) = \left(\frac{-d}{n^\circ}\right)$. For instance $\left(\frac{12}{7}\right) = \left(\frac{-7}{3}\right) = -1$.

For a sequence α with $|\alpha(n)| \le 1$ and $v \ge 1$ let $c_v(n) := \alpha(n)\left(\frac{-v}{n^\circ}\right)$. The mean over $1 \le v \le T^2$ below runs over all integers v. The Pólya–Vinogradov inequality [Dav, Ch. 23] states that $|\sum_{v\le X}\omega(v)| \le C\sqrt{m}\log m$ for every nonprincipal character ω modulo $m \ge 2$ and every X, with C absolute.

Problem 2.2.9. Let $\delta > 0$ and $n \ge 1$, and suppose that n° is not a square. Then $v \mapsto \left(\frac{-v}{n^\circ}\right)$ is ±1 times a nonprincipal character modulo n°, and $\left|\sum_{1\le v\le X}\left(\frac{-v}{n^\circ}\right)\right| \le C_\delta\, n^{1/2+\delta/2}$ for every X, with C_δ depending only on δ.

Problem 2.2.10. Let $\delta > 0$. For $u > 0$, $\sum|w_j(nu)\,w_j(n'u)| \le C_\delta\, u^{-1-2\delta}$, the sum over the pairs $n, n' \ge 1$ with $(nn')^\circ$ a square.

Problem 2.2.11. Let $\delta > 0$, $T > 0$ and $u \ge 1/T$. Then $\sum_{1\le v\le T^2}|\Theta_{c_v}(u)|^2 \le C_\delta\, T^2 u^{-2\delta}$, with C_δ depending only on δ and j, not on α, T or u.

Problem 2.2.12. Let $\delta > 0$. For $T > 0$ and every α with $|\alpha(n)| \le 1$, $\sum_{1\le v\le T^2} J(c_v,T) \le C_\delta\, T^{2+\delta}$, with C_δ depending only on δ and j.

We return to the primes. For a prime $d \equiv 7 \pmod 8$, χ_d is nonprincipal modulo the prime d, hence primitive, and Problem 2.1.6 gives its odd parity.

Problem 2.2.13. Let $\delta > 0$ and $t_0 \geq 0$. For all large X, at most $CX^{1-\delta/2}$ primes $d \leq X$ with $d \equiv 7 \pmod 8$ have a zero of $L(s, \chi_d)$ on $\mathcal{B}(\frac{1}{2} + 9\delta, t_0)$; here C depends on δ and t_0.

Problem 2.2.14. For every $\beta > \frac{1}{2}$ and $t_0 \geq 0$ there are infinitely many primes $d \equiv 7 \pmod 8$ with $L(s, \chi_d) \neq 0$ on $\mathcal{B}(\beta, t_0)$.

Problem 2.2.13 gives a power saving of $X^{-\delta/2}$ relative to the trivial bound X. Density estimates for quadratic twists far stronger than Problem 2.2.13 are known; they rest on mean values of real character sums [Ju], [HB].

Now let ξ run over the characters modulo $N = q_0$, as at the end of Section 2.1. Here $\xi\chi_d$ denotes the character $n \mapsto \xi(n)\left(\frac{n}{d}\right)$ modulo Nd; for an odd prime $d \nmid N$ it is not principal, so $L(s, \xi\chi_d)$ is entire. The smallest case is $N = 8$ and the class $a = 7$.

Problem 2.2.15. Let $d \equiv 7 \pmod 8$ be prime, ξ a character modulo 8, and ξ^* the primitive character inducing ξ, of conductor $f \in \{1, 4, 8\}$. Then $\xi^*\chi_d$ is primitive modulo fd; $L(s, \xi\chi_d) = (1-2^{-s})L(s, \chi_d)$ if ξ is principal, and $L(s, \xi\chi_d) = L(s, \xi^*\chi_d)$ otherwise. Hence for $\beta > 0$ the four twists have no zero on $\mathcal{B}(\beta, t_0)$ exactly when the four $L(s, \xi^*\chi_d)$ have none.

For $d = 7$, each of the four $L(s, \xi\chi_7)$ has positive real part on $3/4 \leq \operatorname{Re} s \leq 1$, $|\operatorname{Im} s| \leq 1/10$, and has no zero in $\mathcal{B}(3/4, 1/10)$; this is a finite computation.

For general N, a character ξ modulo N is induced by a primitive character ξ^* of conductor $M \mid N$. For a prime $d \equiv 7 \pmod 8$ with $d \nmid N$, the twist $\xi^*\chi_d$ is primitive modulo Md and has parity opposite to that of ξ^*. The functions $L(s, \xi\chi_d)$ and $L(s, \xi^*\chi_d)$ have the same zeros in $\operatorname{Re} s > 0$, since the Euler factors removed on passing to level Nd are nonzero there. The following theorems give the exceptional count and the simultaneous choice of d.

Theorem 2.2.1. *Let $N \geq 1$, $\delta > 0$ and $t_0 \in \mathbb{R}$, and let $a \in \mathbb{Z}/N\mathbb{Z}$ be a class such that every prime $p \equiv a \pmod N$ satisfies $p \equiv 7 \pmod 8$ and $\gcd(N, p) = 1$. Call a prime $d \equiv a \pmod N$* bad *if for some character ξ modulo N the function $L(s, \xi\chi_d)$ vanishes at some s with $\operatorname{Re} s \geq \frac{1}{2} + 9\delta$ and $|\operatorname{Im} s| \leq t_0$. Then the number of bad primes $d \leq X$ is $O(X^{1-\delta/2})$ as $X \to \infty$ through the positive integers.*

Theorem 2.2.2. *Let $N \geq 1$, $\beta > \frac{1}{2}$ and $t_0 \in \mathbb{R}$, and let a be a unit of $\mathbb{Z}/N\mathbb{Z}$ such that every prime $p \equiv a \pmod N$ satisfies $p \equiv 7 \pmod 8$ and $\gcd(N, p) = 1$. Then there is a prime $d \equiv a \pmod N$ such that for every character ξ modulo N and every $s \in \mathbb{C}$ with $\operatorname{Re} s \geq \beta$ and $|\operatorname{Im} s| \leq t_0$, $L(s, \xi\chi_d) \neq 0$.*

The hypothesis on the class holds when $8 \mid N$ and $a \equiv 7 \pmod 8$ is a unit; for $1 \leq N < 8$ it holds for no unit class. One d serves all characters ξ and all points of the box at once, and the characters ξ and the class a have the same modulus N.

In the Lean formalization. In the readable edition of the Lean proof [GL], the file `grh/src/Lib/Analytic/QuadraticTwist.lean` contains `New.zero_density_quadTwist` (Theorem 2.2.1) and `New.exists_entireL_quadTwist_ne_zero` (Theorem 2.2.2). The latter states its conclusion for `New.entireL`; the chosen d is odd and prime to N, so `New.quadTwist_ne_one` and the definition of `New.entireL` identify this function with $L(\,\cdot\,, \xi\chi_d)$.

Hints

§2.1.

2.1.1: For odd $p \neq d$ compare $\left(\frac{-d}{p}\right)$ with $\left(\frac{p}{d}\right)$ by reciprocity. For $p = 2$ use $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{-d})/2]$.

2.1.2: Use $\sum_n \Lambda(n)\chi(n)/n = -c_\chi$, convergent by the prime number theorem for χ.

2.1.3: Write S_d as the ramp sum with coefficient 1 minus the one with coefficient χ_d, and use (1) and (3).

2.1.4: The Mellin transform of $G - G_{\chi_d}$ is $F(s) + \frac{L'}{L}(s,\chi_d)/(s(s-1)) - c_{\chi_d}/(s-1)$, with F as in (2). Follow Problem 1.2.4: a real zero of ζ gives the residue sign of that problem, but a real zero of $L(s,\chi_d)$ gives the opposite sign.

2.1.5: As in Problem 2.1.4, with $A(x) := x^\theta - G(x) + G_{\chi_d}(x)$. Show that the abscissa σ_c is less than β: a real singularity $\sigma_c \geq \beta$ would be a real zero of ζ that is not a zero of $L(s,\chi_d)$.

2.1.6: The supplementary laws.

§2.2.

2.2.2: Integrate term by term where the sum converges absolutely. For (i) with $0 < \operatorname{Re} s \leq 1$, bound $\sum_n \chi(n) w_j(nu)$ for small u by partial summation, using that the partial sums of χ are bounded. For the value, $L(1,\chi_7) = \pi/\sqrt{7}$ by the class number formula.

2.2.3: The transformation formula for the theta function of a primitive character [Dav, Ch. 9]; then substitute $u \mapsto 1/(qu)$.

2.2.4: (i) Compare the sum with an integral; w_j is unimodal on $(0,\infty)$. (ii) The difference is the integral over $0 < u \leq 1/T$; use (i) on both integrals.

2.2.5: On $\operatorname{Re} z = \frac{1}{2}$ the factor $u^{z-1/2}$ has modulus 1. Use Problem 2.2.3 and $q^{-1/2} \geq 1/T$.

2.2.6: Evaluate Φ at the given zero s. Apply the three-lines theorem to Φ on $\frac{1}{2} \leq \operatorname{Re} z \leq 2$, with Problems 2.2.4 and 2.2.5 on the two edges. The weight of the right edge at $\operatorname{Re} s$ is $(\operatorname{Re} s - \frac{1}{2})/\frac{3}{2} \geq 6\delta$.

2.2.7: For $t_0 < 0$ the box is empty. For $t_0 \geq 0$, use $L(s,\chi) \neq 0$ for $\operatorname{Re} s \geq 1$ and the positive minimum of $|\Gamma_j|$ on the compact rectangle $\frac{1}{2} \leq \operatorname{Re} s \leq 1$, $|\operatorname{Im} s| \leq t_0$.

2.2.8: Factor $n = 2^k n^\circ$, use $\left(\frac{2}{d}\right) = 1$, and apply reciprocity for the Jacobi symbol with $d \equiv 3 \pmod 4$.

2.2.9: $\left(\frac{-v}{m}\right) = \left(\frac{-1}{m}\right)\left(\frac{v}{m}\right)$. For nontriviality choose v by the Chinese remainder theorem at a prime dividing m to an odd power.

2.2.10: Write $n = 2^a g k^2$ and $n' = 2^b g k'^2$ with g odd and squarefree; the terms with n or n' much larger than $1/u$ are negligible.

2.2.11: Expand the square, so that the inner sum over v is $\sum_v \left(\frac{-v}{(nn')^\circ}\right)$. Bound it by T^2 for the pairs of Problem 2.2.10 and by Problem 2.2.9 for the others, and use $u^{-2} \leq T^2$.

2.2.12: Cauchy–Schwarz over v, Problem 2.2.11 for α and for $\alpha\mu$, then integrate over $u > 1/T$.

2.2.13: Use Problem 2.2.8 with $\alpha = 1$. Take $T = \sqrt{X}$, and compare Problems 2.2.7 and 2.2.12.

2.2.14: Reduce to $\beta < 1$ and put $\delta := (\beta - \frac{1}{2})/9$. If all large primes of the class were bad, partial summation and Problem 2.2.13 would make $\sum \log p/p$ over the class converge.

2.2.15: $\chi_d(2) = 1$. The characters ξ^* and χ_d have coprime conductors, and $\xi^*\chi_d(n) = 0$ for even n when $f > 1$.

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