FROM ZEROS TO A ONE-SIDED BOUND

Prerequisites. The zeta function and Dirichlet L-functions as in [Dav]: orthogonality of characters and $L(1,\chi) \neq 0$ for nonprincipal χ (Ch. 4), the class number formula (Ch. 6), the Hadamard product and the constant B (Ch. 12), nonvanishing on $\operatorname{Re} s = 1$ (Ch. 13 and 14), the explicit formula for Chebyshev's function $\psi(x)$ and its contour shift (Ch. 17), their analogues for $L(s,\chi)$ (Ch. 12 and 19), and the prime number theorem in progressions to a fixed modulus (Ch. 20). Chebyshev's bound $\psi(x) \ll x$ [MV, §2.2]. Mellin transforms. Landau's theorem is stated in §1.2. Section 1.4 uses the auxiliary-prime theorem stated as Theorem 2.2.2 in Chapter 2. Chapter 16 is used only for motivation, in §1.3.

A comparison of divisibility with size, as in Chebyshev's method, gives upper bounds for weighted sums over primes. Can an upper bound for a sum over primes prove the Riemann hypothesis? This chapter shows that it can, for a suitable weight. After the zeta function, Theorem 1.4.1 gives the criterion for Dirichlet L-functions in the form used in the proof. The upper bound itself is not treated here.

1.1. The ramp sum

Write $v_+ := \max(v, 0)$ for real v, and let Λ be the von Mangoldt function. The *ramp* at x is the function $n \mapsto (x/n - 1)_+$: it is large for small n, decreases, and vanishes for $n \geq x$. Let

$$S(x) := \sum_n \Lambda(n) \Big(\frac{x}{n} - 1\Big)_+, \qquad G(x) := S(x) - x \log x + (1+\gamma)x, \tag{1}$$

where γ is Euler's constant, and let $\psi(x) := \sum_{n \leq x} \Lambda(n)$ be Chebyshev's function. In §1.4, ψ denotes a Dirichlet character. The two subtracted terms are the main term of S, as the problems show.

Problem 1.1.1. For $n \geq 1$ and $\operatorname{Re} s > 1$, $\int_1^\infty (x/n - 1)_+ \, x^{-s-1} \, dx = n^{-s} / \big(s(s-1)\big)$.

Problem 1.1.2. $S(x) = x \int_1^x \psi(t) \, t^{-2} \, dt$. Hence $G(x) \ll x \log x$ for $x \geq 2$.

Problem 1.1.3. For $\operatorname{Re} s > 1$,

$$F(s) := \int_1^\infty G(x) \, x^{-s-1} \, dx = -\frac{\zeta'}{\zeta}(s) \, \frac{1}{s(s-1)} - \frac{1}{(s-1)^2} + \frac{1+\gamma}{s-1}. \tag{2}$$

Problem 1.1.4. F is meromorphic in $\mathbb{C}$ and regular at $s = 1$. Its poles in $\operatorname{Re} s > 0$ are the zeros ρ of ζ in the critical strip $0 \leq \operatorname{Re} s \leq 1$, and a zero of multiplicity m gives residue $-m/\big(\rho(\rho-1)\big)$.

Problem 1.1.5. The residue of $F(s)x^s$ at $s = 0$ is $\log 2\pi$.

Problems 1.1.3–1.1.5 and a contour shift as in [Dav, Ch. 17], applied to $-\frac{\zeta'}{\zeta}(s) \, x^s/(s(s-1))$, the part of $F(s)x^s$ that comes from S, where now every integral converges absolutely because of the factor $1/(s(s-1))$, give the following.

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Theorem 1.1.1. *For $x > 1$,*

$$G(x) = \log 2\pi - \sum_\rho \frac{x^\rho}{\rho(\rho-1)} + O(x^{-2}),$$

where ρ runs over the zeros of ζ in the critical strip with multiplicity, and the sum converges absolutely.

Problem 1.1.6. $\sum_\rho 1/\big(\rho(1-\rho)\big) = 2 + \gamma - \log 4\pi = 0.0461\ldots$, with multiplicity.

Problem 1.1.7. If the Riemann hypothesis holds, then $|G(x) - \log 2\pi| \le (2 + \gamma - \log 4\pi)\sqrt{x} + O(x^{-2})$.

For instance $G(10^4) \approx 1.749$, $G(10^5) \approx 3.671$, $G(10^6) \approx 11.040$ and $G(2 \cdot 10^6) \approx -12.268$, to be compared with $\log 2\pi \approx 1.838$ and $0.0462\sqrt{10^6} = 46.2$.

Problem 1.1.8. Let $\rho = \beta + i\gamma_0$ be a nonreal zero of ζ, and let $c := -1/\big(\rho(\rho-1)\big)$. The joint term $cx^\rho + \bar{c}x^{\bar\rho}$ of ρ and $\bar\rho$ in Theorem 1.1.1 exceeds $|c|x^\beta$ for all x in a set whose image under log has density $1/3$ in $[0,\infty)$, and it is less than $-|c|x^\beta$ on another such set.

If $\beta > \frac12$, the contribution of this conjugate pair exceeds $\sqrt{x}$ along one sequence tending to infinity and falls below $-\sqrt{x}$ along another. A comparison of divisibility with size sees only one sign. The next section shows that one is enough.

1.2. One side suffices

We use the following measurable version of Landau's theorem [Lan]. Montgomery and Vaughan [MV, Lemma 15.1] assume local Riemann integrability and use x^{-s}; their proof applies to the Lebesgue integral below after replacing s by $s+1$. Let $A\colon [1,\infty) \to \mathbb{R}$ be measurable and bounded on compact sets, with $A(x) \ge 0$ for $x \ge X_0$. Let σ_c be the infimum of the real σ with $\int_1^\infty |A(x)| x^{-\sigma-1}\,dx < \infty$, and suppose that $-\infty < \sigma_c < \infty$. Then $H(s) := \int_1^\infty A(x) x^{-s-1}\,dx$ is holomorphic in $\operatorname{Re} s > \sigma_c$, and it has no holomorphic continuation to a neighborhood of σ_c.

Problem 1.2.1. In this situation $H(\sigma)$ is bounded below for real $\sigma \in (\sigma_c, \sigma_c + 1]$.

Fix $0 < \theta < 1$, suppose that $G(x) \le x^\theta$ for all large x, and let $A(x) := x^\theta - G(x)$.

Problem 1.2.2. $-\infty < \sigma_c \le 1$, and $H(s) = 1/(s-\theta) - F(s)$ for $\operatorname{Re} s > 1$.

Problem 1.2.3. $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.

Problem 1.2.4. If $\sigma_c > \theta$, then σ_c is a real zero of ζ, and $H(\sigma) \to -\infty$ as $\sigma \downarrow \sigma_c$. Hence $\sigma_c \le \theta$.

By Problem 1.2.3 ζ has no real zero in $(0,1)$, but Problem 1.2.4 does not use this, and in §1.4 the same argument will be applied where real zeros cannot be excluded in advance. By Problem 1.2.4, $F = 1/(s-\theta) - H$ is holomorphic in $\operatorname{Re} s > \theta$, and by Problem 1.1.4 this excludes zeros there.

Theorem 1.2.1. *Let $0 < \theta < 1$. If $G(x) \le x^\theta$ for all large x, then ζ has no zeros in $\operatorname{Re} s > \theta$.*

Problem 1.2.5. If for every $\varepsilon > 0$ one has $G(x) \le x^{1/2+\varepsilon}$ for all large x, then the Riemann hypothesis holds.

The transform F continues meromorphically, with at most simple poles in $\operatorname{Re} s > 0$. This, the sign of its residues at real points to the right of θ, and the eventual upper bound for G are what the argument of Problem 1.2.4 uses. In the variable $v = \log x$, with $E(v) := e^{-v}G(e^v)$, the transform $F(s)$ becomes $\int_0^\infty E(v)e^{(1-s)v}\,dv$, and the bound $G(x) \le x^\theta$ becomes $E(v) \le e^{(\theta-1)v}$. In this form the argument proves the following.

Theorem 1.2.2. *Let $E\colon \mathbb{R} \to \mathbb{R}$ and $r\colon \mathbb{C} \to \mathbb{C}$. Suppose that E is Borel measurable, and that there are $C \in \mathbb{R}$ and a meromorphic function Φ on $\mathbb{C}$ such that $|E(v)| \le Ce^v$ for all $v \ge 0$, such that $\Phi(s) = \int_0^\infty E(v)e^{(1-s)v}\,dv$ for all s with $\operatorname{Re} s > 2$, and such that $(z-\rho)\Phi(z) \to r(\rho)$ as $z \to \rho$, $z \ne \rho$, for every ρ with $\operatorname{Re}\rho > 0$. Let $\beta > 0$. Suppose that $\operatorname{Re} r(x) \ge 0$ for every real $x > \beta$, and that there is V with $E(v) \le e^{(\beta-1)v}$ for all $v \ge V$. Then $r(\rho) = 0$ for every $\rho \in \mathbb{C}$ with $\operatorname{Re}\rho > \beta$.*

The hypothesis on Φ says that its poles in $\operatorname{Re} s > 0$ are simple, with residues r; the conclusion says that it has none in $\operatorname{Re} s > \beta$. Only a sign is assumed at real points, and only there.

1.3. Smoothing, and the triangle weight

Why the ramp, and not another weight? Let $\phi\colon (0,\infty) \to \mathbb{R}$ be measurable, vanish on $(1,\infty)$, be bounded on compact subsets of $(0,1]$, and satisfy $\phi(t) = O(1/t)$ as $t \to 0$. Let

$$\psi_\phi(x) := \sum_n \Lambda(n)\phi(n/x), \qquad \tilde\phi(s) := \int_0^\infty \phi(t)t^{s-1}\,dt \quad (\operatorname{Re} s > 1).$$

We use $\tilde\phi$ on its half-plane of absolute convergence and, for the three weights below, also for the meromorphic continuation given by their rational formulas. The sharp cutoff $\phi = \mathbf{1}_{(0,1]}$ gives $\psi_\phi = \psi$ and $\tilde\phi(s) = 1/s$. The ramp is $\phi(t) = (1/t - 1)_+$, with $\tilde\phi(s) = 1/(s(s-1))$ by Problem 1.1.1. The *triangle weight* is $\phi(t) = (1-t)_+$, with $\psi_\phi(x) = x^{-1}\int_0^x \psi(t)\,dt$.

Problem 1.3.1. For $\operatorname{Re} s > 1$, $\int_0^\infty \psi_\phi(x)x^{-s-1}\,dx = -\frac{\zeta'}{\zeta}(s)\,\tilde\phi(s)$.

Problem 1.3.2. For the triangle weight $\tilde\phi(s) = 1/(s(s+1))$ for $\operatorname{Re} s > 0$. If $\phi_1(t) := \phi(t)/t$, then $\tilde\phi_1(s) = \tilde\phi(s-1)$ wherever either integral converges absolutely; for the triangle weight ϕ_1 is the ramp.

The following theorem is proved as Theorem 1.1.1: by Problem 1.3.1 with $\tilde\phi(s) = 1/(s(s+1))$, shift the line of integration to $\operatorname{Re} s = -3/2$ as in [Dav, Ch. 17], along horizontal segments at heights T where $\zeta'/\zeta \ll \log^2 T$. The factor $1/(s(s+1))$ makes every integral converge absolutely, and the poles of $-\frac{\zeta'}{\zeta}(s)\,x^s/(s(s+1))$ to the right of $-3/2$ are at 1, 0, -1 and the zeros ρ.

Theorem 1.3.1. *For $x > 1$,*

$$\sum_{n\le x}\Lambda(n)\Big(1-\frac{n}{x}\Big) = \frac{x}{2} - \log 2\pi - \sum_\rho \frac{x^\rho}{\rho(\rho+1)} + O\Big(\frac{1}{x}\Big).$$

As a check, the left side minus $(x/2 - \log 2\pi)$ is approximately 0.106, 7.975 and -12.115 at $x = 10^4$, 10^6 and $2\cdot 10^6$. For the ramp and the triangle weight

$|\tilde{\phi}(\rho)| \ll |\rho|^{-2}$, so the sums over zeros converge absolutely. For the sharp cutoff they do not, since $\sum_\rho 1/|\rho|$ diverges.

Problem 1.3.3. Let $0 < \theta < 1$. If $\sum_{n\le x} \Lambda(n)(1 - n/x) \le x/2 + x^\theta$ for all large x, then ζ has no zeros in $\operatorname{Re} s > \theta$.

We return to real zeros in §1.4.

The ramp itself is forced by the arithmetic. Theorem 16.3.1 gives, at primes $p \equiv 3 \pmod 8$, lower bounds for p-adic valuations of the form $\sum_{i \text{ odd}} (\lfloor a/p^i \rfloor - w)_+$. Up to an error of 1 for each odd i with $p^i \le a$, and with $a = wx$, the summand is $(a/p^i - w)_+ = w(x/p^i - 1)_+$, which is w times the ramp at x. Bounds of this form at every inert prime, summed with weights $\log p$, give the ramp sum over odd powers of primes inert in an imaginary quadratic field; Chapter 2 says what this sum counts.

1.4. Dirichlet L-functions

What replaces G for the L-functions of the characters modulo q? The arithmetic bounds sums over primes that do not split in an imaginary quadratic field. For a prime $d \equiv 3 \pmod 4$, the Legendre symbol $\chi_d(n) = (n/d)$ is the character of $K = \mathbb{Q}(\sqrt{-d})$, and at a prime p the weight $1 - \chi_d(p)$ is 2 if p is inert, 1 if $p = d$, and 0 if p splits. This weight brings in the twisted L-functions $L(s, \psi\chi_d)$, whose orders of vanishing are subtracted from the untwisted ones in the residues below. The prime d will be chosen so that they do not vanish where it matters.

For a primitive nonprincipal Dirichlet character χ let $c_\chi := \frac{L'}{L}(1, \chi)$ and

$$G_\chi(x) := \sum_{n\le x} \Lambda(n)\chi(n)\Big(\frac{x}{n} - 1\Big) + c_\chi x. \tag{3}$$

Let d be an odd prime, $\chi_d(n) := (n/d)$ the Legendre symbol, and $q \ge 1$ an integer prime to d. Let φ be Euler's function. For $n \ge 1$ let

$$\lambda(n) := \Lambda(n)\big(1 - \chi_d(n)\big) \text{ if } n \equiv 1 \pmod q, \qquad \lambda(n) := 0 \text{ otherwise.}$$

The analytic definitions below apply to every odd prime d. For a Dirichlet character ψ modulo q, let $L(s, \psi)$ be the meromorphic continuation of $\sum_{n\ge1} \psi(n)n^{-s}$ from $\operatorname{Re} s > 1$; its Euler product there runs over the primes $p \nmid q$. Let $E_\psi := L(\,\cdot\,, \psi)$ if ψ is not principal, and if ψ is principal let E_ψ be the entire continuation of $(s-1)L(s, \psi)$, with $E_\psi(1) = \prod_{p|q}(1 - p^{-1})$. Let $m_\psi(\rho) \ge 0$ be the order of vanishing of E_ψ at ρ. The product $\psi\chi_d$ is a character modulo qd; it is not principal, and we write $E_{\psi\chi_d} := L(\,\cdot\,, \psi\chi_d)$ and $m_{\psi\chi_d}$ for its order. Let

$$G_{d,q}(s) := \frac{1}{\varphi(q)}\Big(\sum_\psi \Big(\frac{E'_{\psi\chi_d}}{E_{\psi\chi_d}}(s) - \frac{E'_\psi}{E_\psi}(s)\Big) - 1\Big),$$

the sum over the characters ψ modulo q. The E's do not vanish on $\operatorname{Re} s \ge 1$, so $G_{d,q}$ is holomorphic there. Let $c_{d,q} := \operatorname{Re} G_{d,q}(1)$, and for real $x > 0$ let

$$R_{d,q}(x) := \sum_{1\le n\le x} \lambda(n)\Big(\frac{1}{n} - \frac{1}{x}\Big) - \frac{\log x}{\varphi(q)} - c_{d,q}. \tag{4}$$

The simplest case is $q = 1$, where $\lambda(n) = \Lambda(n)(1 - \chi_d(n))$ for all n and $E_{\mathbf{1}}(s) = (s-1)\zeta(s)$ for $s \ne 1$, with $E_{\mathbf{1}}(1) = 1$.

Problem 1.4.1. $\lambda(n) = \varphi(q)^{-1}\sum_{\psi}\big(\psi(n) - \psi(n)\chi_d(n)\big)\Lambda(n)$, and $0 \le \lambda(n) \le 2\Lambda(n)$. For $q = 1$ and $d = 7$ the nonzero values for $n \le 11$ are $\lambda(3) = 2\log 3$, $\lambda(5) = 2\log 5$ and $\lambda(7) = \log 7$.

Problem 1.4.2. For $\operatorname{Re} s > 1$, $\sum_{n\ge1}\lambda(n)n^{-s} = s/\big(\varphi(q)(s-1)\big) + G_{d,q}(s)$.

Problem 1.4.3. For $\operatorname{Re} s > 1$, $\int_1^\infty R_{d,q}(y)\, y^{-s}\, dy = \big(G_{d,q}(s)/s - c_{d,q}\big)/(s-1)$.

Problem 1.4.4. (i) The function $\Phi(s) := \big(G_{d,q}(s)/s - c_{d,q}\big)/(s-1)$ is meromorphic on $\mathbb{C}$ and regular at $s = 1$. At every $\rho \ne 0, 1$ it has at most a simple pole, with residue

$$\frac{1}{\varphi(q)\,\rho(1-\rho)}\sum_{\psi}\big(m_\psi(\rho) - m_{\psi\chi_d}(\rho)\big).$$

(ii) If no $L(s, \psi\chi_d)$ vanishes at a real $x \in (0, 1)$, the residue at x is ≥ 0, and it is > 0 if some $L(s, \psi)$ vanishes there. A zero of order m of one $L(s, \psi\chi_d)$ at a point $\rho \ne 0, 1$ where no other E vanishes gives the residue $-m/\big(\varphi(q)\rho(1-\rho)\big)$.

At $\rho \ne 0, 1$ the residue depends on the difference between the total untwisted and twisted orders. A common zero can therefore reduce or cancel the contribution of an untwisted factor, and at a real point $0 < \rho < 1$ a twisted zero alone gives a negative residue. Neither happens where no twist vanishes.

Problem 1.4.5. $c_{d,1} = -1 - \gamma + \frac{L'}{L}(1, \chi_d)$. For $d = 7$, $L(1, \chi_7) = \pi/\sqrt{7} = 1.1874104\ldots$; with $\frac{L'}{L}(1, \chi_7) \approx 0.0156357$ this gives $c_{7,1} \approx -1.56158$.

Problem 1.4.6. For $y \ge 1$, $yR_{d,1}(y) = G(y) - G_{\chi_d}(y)$, with G as in (1) and G_{χ_d} as in (3).

For instance $R_{7,1}(y) \approx 6.27 \cdot 10^{-4}$, $1.05 \cdot 10^{-4}$ and $-7.71 \cdot 10^{-5}$ at $y = 10^4$, 10^5 and 10^6.

Problem 1.4.7. (i) $\sum_{n\le y}\lambda(n)(1/n - 1/y) - \log y/\varphi(q)$ has a limit as $y \to \infty$. (ii) The limit is $c_{d,q}$; that is, $R_{d,q}(y) \to 0$.

So $c_{d,q}$ is the constant that makes the remainder small. The proof gives more: for every odd prime d, every $q \ge 1$ prime to d and every integer $B \ge 0$, $|R_{d,q}(y)| \le (\log y)^{-B}$ for all large real y. It does this by a Tauberian theorem applied to $v \mapsto R_{d,q}(e^v)$ and the transform identity of Problem 1.4.3 for $\operatorname{Re} s > 2$.

Now fix a point ρ. The next problem is the analogue of Theorem 1.2.1 for the $L(s, \psi)$, at one prime d whose twists do not vanish near the real axis up to the height of ρ.

Problem 1.4.8. Let $d \equiv 7 \pmod 8$ be prime, $q \ge 1$ prime to d, $\beta > 0$, and $\rho \ne 1$ with $\operatorname{Re}\rho > \beta$. Suppose that $L(s, \psi\chi_d) \ne 0$ for every ψ modulo q and every s with $\operatorname{Re} s \ge \beta$ and $|\operatorname{Im} s| \le |\operatorname{Im}\rho|$, and that $R_{d,q}(y) \le y^{\beta-1}$ for all large real y. Then $L(\rho, \psi) \ne 0$ for every ψ modulo q.

Problem 1.4.8 assumes no separate exclusion of real zeros of the untwisted L-functions. For $\beta > 1/2$, Theorem 2.2.2 of Chapter 2 supplies the required twist nonvanishing for some prime d in any residue class satisfying its hypotheses. For a Dirichlet character ξ modulo N and an odd prime d not dividing N, write $\xi\chi_d$ for their product, a character modulo Nd. That theorem says: let $N \ge 1$, $\beta > 1/2$ and $t_0 \in \mathbb{R}$, and let a be a unit modulo N such that every prime $p \equiv a \pmod N$

satisfies $p \equiv 7 \pmod 8$ and $\gcd(N,p)=1$; then there is a prime $d \equiv a \pmod N$ such that $L(s,\xi\chi_d) \neq 0$ for every character ξ modulo N and every s with $\operatorname{Re} s \geq \beta$ and $|\operatorname{Im} s| \leq t_0$. It says nothing about $R_{d,q}$.

Let q_0 be a positive multiple of 8, and let $d_0 \equiv 7 \pmod 8$ be prime to q_0. Every prime $d \equiv d_0 \pmod{q_0}$ is then $\equiv 7 \pmod 8$ and prime to q_0. The next theorem assumes the bound for R_{d,q_0} for every prime d in this class. Given ρ, Theorem 2.2.2 with $N = q_0$, $a = d_0$ and $t_0 = |\operatorname{Im}\rho|$ chooses d in the class, and Problem 1.4.8 applies to it.

Theorem 1.4.1. *Let $q_0 \geq 1$ with $8 \mid q_0$, let $d_0 \geq 0$ be an integer with $d_0 \equiv 7 \pmod 8$ and $\gcd(d_0,q_0)=1$, and let $\beta > 1/2$. Suppose that for every prime $d \equiv d_0 \pmod{q_0}$ there is $Y \in \mathbb{R}$ such that $R_{d,q_0}(x) \leq x^{\beta-1}$ for every real $x \geq Y$. Then $L(\rho,\psi) \neq 0$ for every character ψ modulo q_0 and every $\rho \neq 1$ with $\operatorname{Re}\rho > \beta$.*

The hypothesis concerns every prime of one class modulo q_0, and the point Y from which the bound holds may depend on d. For $q = 1$, by Problem 1.4.6, a bound $R_{d,1}(x) \leq x^{\theta-1}$ reads $G(x) - G_{\chi_d}(x) \leq x^{\theta}$; compare Theorem 1.2.1.

Problem 1.4.9. Let $M \geq 1$. If for every $\beta > 1/2$ the hypothesis of Theorem 1.4.1 holds for $q_0 = 8M$ and $d_0 = 8M-1$, then $L(s,\chi) \neq 0$ for every character χ modulo M and every $s \neq 1$ with $\operatorname{Re} s > 1/2$.

In the Lean formalization. In the Lean proof [GL] (readable edition), Theorem 1.4.1 is `New.LFunction_ne_zero_of_primeRemainder_le` in `New/Reduction/Zeros/Criterion.lean`; for $x > 0$, $R_{d,q}(x)$ is `New.primeRemainder` at d, q, x in `New/Reduction/Zeros/PrimeError.lean`. Theorem 1.2.2 is `New.IsPolar.landau` in `Lib/Analytic/Landau.lean`, with the definition `New.IsPolar` written out. The bound $|R_{d,q}(y)| \leq (\log y)^{-B}$ is `New.eventually_abs_primeRemainder_le` in `New/Reduction/Zeros/PrimeNumberTheorem.lean`.

Hints

§1.1.

1.1.2: Integrate each step of ψ. For the bound use Chebyshev's estimate.

1.1.3: Use Problem 1.1.1 termwise, and compute the transforms of $x\log x$ and x.

1.1.4: Use $-\zeta'/\zeta(s) = 1/(s-1) - \gamma + O(s-1)$ and expand $1/(s(s-1))$ at $s=1$.

1.1.5: The functional equation gives $\zeta'/\zeta(0) = \log 2\pi$.

1.1.6: Write $1/(\rho(1-\rho)) = 1/\rho + 1/(1-\rho)$ and use the value of B in [Dav, Ch. 12].

§1.2.

1.2.1: Split the integral at X_0.

1.2.2: Use Problem 1.1.2 for $\sigma_c \leq 1$ and Problem 1.1.3 for the formula for H. For $\sigma_c > -\infty$ use Problem 1.1.4: ζ has infinitely many zeros in the critical strip [Dav, Ch. 12], so it has nonreal ones.

1.2.3: Consider $(1-2^{1-\sigma})\zeta(\sigma) = \sum_{n\geq 1}(-1)^{n-1}n^{-\sigma}$.

1.2.4: By Landau's theorem σ_c is a singularity of H. Of the real points in $(\theta,1]$, only real zeros of ζ can be singular (Problem 1.1.4). Find the sign of the residue of F at a real zero $\beta \in (0,1)$, and compare with Problem 1.2.1.

1.2.5: Use Theorem 1.2.1 and the symmetry $\rho \mapsto 1-\rho$.

§1.3.

1.3.1: Substitute $x = n/t$ in each term.

1.3.3: Repeat §1.2 with $\tilde{\phi}(s) = 1/(s(s+1))$, including the argument for $\sigma_c > -\infty$ in Problem 1.2.2, and compare the sign of the residue at a real zero with that in Problem 1.2.4; conclude with Problem 1.2.3.

§1.4.

1.4.1: Orthogonality of the characters modulo q; a number $n \equiv 1 \pmod q$ is prime to q.

1.4.2: Problem 1.4.1 and $-\frac{L'}{L}(s,\chi) = \sum_n \chi(n)\Lambda(n)n^{-s}$; for the principal ψ, E'_ψ/E_ψ and L'/L differ by $1/(s-1)$.

1.4.3: As Problem 1.1.3: $(1/n - 1/y)_+ y^{-s} = (y/n-1)_+ y^{-s-1}$, so Problem 1.1.1 applies; then Problem 1.4.2.

1.4.4: (i) A zero of order m of an entire function at ρ is a simple pole of its logarithmic derivative, with residue m. At $s=1$: the characters modulo q are closed under complex conjugation, so $G_{d,q}(1)$ is real.

1.4.5: For $q=1$ the untwisted factor is $(s-1)\zeta(s) = 1 + \gamma(s-1) + \cdots$. For $d=7$ use the class number formula.

1.4.6: Problem 1.4.5; for $n \le y$ the ramp is $y(1/n - 1/y)$.

1.4.7: (i) Partial summation, as in Problem 1.1.2, with the prime number theorem with error term for the characters modulo qd [Dav, Ch. 20]. (ii) If $R_{d,q} \to \ell$, compare the behaviour of the two sides of Problem 1.4.3 as $s \downarrow 1$, using Problem 1.4.4.

1.4.8: Theorem 1.2.2 for $E(v) := R_{d,q}(e^v)$, with Problems 1.4.3 and 1.4.4. Check the sign of the residue at real x in $(\beta, 1)$, at 1 and in $(1,\infty)$ separately.

1.4.9: A character modulo M induces one modulo $8M$; the two L-functions differ by finitely many Euler factors.

References

[Dav] H. Davenport, *Multiplicative Number Theory*, 3rd ed., revised by H. L. Montgomery, Grad. Texts in Math. **74**, Springer, 2000.

[GL] L. Sernau, *GRH for Dirichlet L-functions, in Lean 4*, a shorter rewriting of the Lean proof, text by Claude, `https://github.com/lsernau/grh-lean`, commit `72fead1`, readable edition `grh/`.

[Lan] E. Landau, *Über einen Satz von Tschebyschef*, Math. Ann. **61** (1905), 527–550.

[MV] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory*, Cambridge Stud. Adv. Math. **97**, Cambridge University Press, 2007.